Introduction to Feynman integrals and multiloop techniques

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The discovery of the Higgs boson has heralded the era of precision in hadron collider physics. Disentangling potential new physics effects from the wealth of data requires a very high level of control over theoretical predictions for Standard Model cross sections which is very often limited by our ability to compute complicated Feynman diagrams. Feynman integrals are a rapidly developing field and there are many competing methods which each have their own merits and limitations and state-of-the art problems often require a combinations of various tools. This course provides an introduction to some of the most widely used techniques with the aim of providing a starting point on how to tackle simple and more complicated calculations.

I will start by reviewing the basic concepts of dimensional regularization and Feynman parametrization and then move to more advanced topics including sector decomposition, Mellin Barnes representations, reduction to master integrals using integration-by-parts identities, solving master integrals by differential equations and the expansion by regions. The application of all these techniques is illustrated by explicit examples.

Contents

1 Introduction

In these lecture notes we give an introduction to the very wide and active field of Feynman integrals and the techniques used to evaluate them. We assume familiarity with the basic ideas of perturbative quantum field theory and Feynman diagrams, but introduce all of the concepts that are used in the example calculations below. In large parts these notes are inspired by the book [1] by Vladimir A. Smirnov, but some more recent development concerning e.g. the method of differential equations are discussed as well.

We will focus almost entirely on scalar integrals

$$F(\{n_i\}) = \int \frac{d^d l_1}{i\pi^{d/2}} \cdots \frac{d^d l_L}{i\pi^{d/2}} \frac{1}{[P_1^2 - m_1^2 + i0]^{n_1} \dots [P_N^2 - m_N^2 + i0]^{n_N}}\,, \tag{1.1}$$

where P_i are linear combinations of the loop momenta l_j and the external momenta p_k. The exponents n_i are commonly called indices and can be negative when scalar products appear in the numerator. We frequently use the shorthands

$$[dl] = \frac{d^d l}{i\pi^{d/2}}\,, \qquad\qquad [dl]_L = \prod_{j=1}^{L} \frac{d^d l_j}{i\pi^{d/2}}\,. \tag{1.2}$$

When dealing with effective field theories (EFT) we often encounter propagators with a linear dependence on the loop momenta $1/(n \cdot l)$ or $1/(v \cdot l + \omega)$ where n, v are light-like ($n^2 = 0$) or time-like ($v^2 = 1$) reference vectors and we will also consider some examples involving propagators like this. There are also Glauber modes with propagators of the form $1/l_\perp^2$ and in non-relativistic effective theories propagators of the type $1/(l^0 - \mathbf{l}^2/(2m))$ appear, but they will not be considered here. Tensor integrals are briefly discussed in Section 2.4 and can typically be avoided, which is why we will not consider them further.

In Section 2 we discuss the concept of dimensional regularization of UV and IR divergences and introduce Feynman parametrization which is applied to several examples. Afterwards, we consider methods that start from the Feynman parametrization of a given integral and allow its numerical or analytic evaluation in more complicated cases – sector decomposition in Section 3 and Mellin-Barnes representations in Section 4. Section 5 deals with the reduction of the large number of integrals that appear in scattering amplitudes to a set of master integrals through the use of integration-by-parts (IBP) identities. IBP identities allow to write linear systems of differential equations for the master integrals whose solution is discussed in Section 6. Last but not least we discuss the expansion by regions, a method that allows to determine integrals as an expansion in a small parameter, in Section 7. We summarize and provide references to additional literature in Section 8. At the end of each section we list the names of the `Mathematica` files containing examples or solutions.

2 Basics

2.1 Divergences and dimensional regularization

While physical observables must be well-defined, individual diagrams and intermediate expressions typically diverge in certain limits of the loop momenta. We will only discuss regularization of divergences not their cancellation in physical quantities which requires renormalization and the combination of virtual corrections with real-emission ones. Under regularization we understand the introduction of some auxiliary parameter in a divergent integral which rendes the integral well-defined. This is of course not unique. Examples are a cutoff

$$\int_0^\infty \frac{dx}{1+x} \to \int_0^\Lambda \frac{dx}{1+x} = \ln(1+\Lambda)\,, \tag{2.1}$$

or an analytic regulator

$$\int_0^\infty \frac{dx}{1+x} \to \int_0^\infty \frac{dx}{(1+x)^{1+\alpha}} = \frac{1}{\alpha}\,. \tag{2.2}$$

Clearly the result depends on the regularization procedure, but if we consider a finite quantity and apply the same regularization procedure to all parts the regulator dependence cancels in the limit where the regulator is removed and we obtain a unique result

$$\int_0^\infty \frac{dx}{1+x} - \int_0^\infty \frac{dx}{2+x} = \begin{cases} \ln(1+\Lambda) - \ln\left(1+\frac{\Lambda}{2}\right) \overset{\Lambda\to\infty}{=} \ln(2), & \text{cutoff regulator,} \\ \frac{1}{\alpha} - \frac{2^{-\alpha}}{\alpha} \overset{\alpha\to 0}{=} \ln(2), & \text{analytic regulator.} \end{cases} \tag{2.3}$$

The established regularization procedure for loop integrals is dimensional regularization which respects the symmetries of the theory, i.e. Lorentz and gauage invariance. It also has some particularly useful properties for effective field theories. Let's introduce dimensional regularization by looking at some examples for the most common types of divergences. We can imagine performing the loop integral not in 4 but in d space-time dimensions which means $d-1$ space dimensions. At this point we think of d as an integer larger than one. Firstly consider the massive tadpole integral with index two

$$\begin{aligned}\int \frac{d^d l}{i\pi^{d/2}} \frac{1}{[l^2 - m^2 + i0]^2} &= \int \frac{d^d l_E}{\pi^{d/2}} \frac{1}{[l_E^2 + m^2 - i0]^2} \\ &= \int \frac{d\Omega_d}{\pi^{d/2}} \int_0^\infty d|l_E| \frac{|l_E|^{d-1}}{[|l_E|^2 + m^2 - i0]^2} \\ &= \frac{2}{\Gamma(d/2)} \int_0^\infty d|l_E| \frac{|l_E|^{d-1}}{[|l_E|^2 + m^2 - i0]^2}\,. \end{aligned} \tag{2.4}$$

We have performed the Wick rotation $l^0 \to i l_E^0$ and factorized the resulting Euclidean integral into an angular and radial integration. The angular integral $\int d\Omega_d$ has been determined by solving the d-dimensional Gauss integral two different ways:

$$\int_0^\infty d^d\mathbf{x} e^{-\mathbf{x}^2} = \left(\int_0^\infty dx e^{-x^2}\right)^d = \pi^{d/2}$$

$$
\begin{aligned}
&= \int d\Omega_d \int_0^\infty d|\mathbf{x}||\mathbf{x}|^{d-1} e^{-|\mathbf{x}|^2} = \int d\Omega_d \int_0^\infty \frac{du}{2} u^{d/2-1} e^{-u} \\
&= \frac{\Gamma(d/2)}{2} \int d\Omega_d \,. \qquad (2.5)
\end{aligned}
$$

The Euler Gamma function is defined as

$$
\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x} \qquad (2.6)
$$

for $\mathrm{Re}(z) > 0$ and extends the factorial to which it relates by $n! = \Gamma(n+1)$ for natural numbers n. It can be analytically continued in the entire complex plane and exhibits single poles at non-positive integer arguments. It satisfies the relation $\Gamma(z+1) = z\Gamma(z)$. For large $|l_E|$ the integrand in (2.4) behaves as $|l_E|^{d-5}$ and the integral is divergent for $d \geq 4$. This type of divergence is called a *UV divergence* because it is caused by the region of large momenta. The expression in the last line of (2.4) now *defines* the value of the original dimensionally regularized integral for complex values of d. Assuming $\mathrm{Re}(d) < 4$ and changing the integration variable to $x = |l_E|^2/m^2$ we can proceed with the calculation

$$
\begin{aligned}
(2.4) =& \frac{2}{\Gamma(d/2)} \left(m^2 - i0\right)^{d/2-2} \int_0^\infty dx \frac{x^{d/2-1}}{2(1+x)^2} \\
=& \frac{1}{\Gamma(d/2)} \left(m^2 - i0\right)^{d/2-2} \frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)} \\
=& \Gamma(2-d/2) \left(m^2 - i0\right)^{d/2-2} . \qquad (2.7)
\end{aligned}
$$

The x integration has been evaluated in terms of the Euler Beta function

$$
B(a,b) \equiv \int_0^\infty dx \frac{x^{a-1}}{(1+x)^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \,. \qquad (2.8)
$$

The result (2.7) can now be analytically continued to arbitrary values of d in the complex plane and has single poles at $d = 4, 6, 8, \ldots$ corresponding to negative integer arguments of the Gamma function. Setting $d = 4 - 2\epsilon$ we can expand around four space-time dimensions and obtain the dimensionally regularized result

$$
(2.4) = \frac{1}{\epsilon} - \gamma_E - \ln(m^2) + \mathcal{O}(\epsilon) \,, \qquad (2.9)
$$

where $\gamma_E = 0.577216\ldots$ is Euler's constant and the pole in ϵ is called a UV pole because of the UV nature of the divergence in the loop integral.

As a second example we take the massless scalar vertex shown in Figure 1 with the on-shell condition $p^2 = 0 = p'^2 \equiv (p+q)^2$

$$
\begin{aligned}
& \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{[(l+p)^2 + i0][(l+p')^2 + i0][l^2 + i0]} \\
=& \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{[l^2 + 2l \cdot p + i0][l^2 + 2l \cdot p' + i0][l^2 + i0]} \,. \qquad (2.10)
\end{aligned}
$$

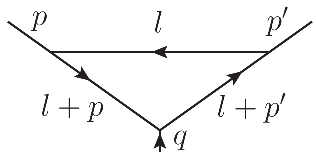


Figure 1: One-loop vertex diagram.

This integral is UV finite for $d < 6$ since the radial integrand after Wick rotation behaves like $d|l_E||l_E|^{d-7}$ for $|l_E| \to \infty$. On the other hand, for $|l_E| \to 0$ the integrand scales like $d|l_E||l_E|^{d-5}$ and thus diverges for $d \leq 4$ which is called a *soft divergence*. This divergence is only present when the external momenta are on-shell, because then the internal propagators become on-shell in the soft limit $(l+p)^2 \xrightarrow{l \to 0} p^2 = 0$, and also manifests as a $1/\epsilon$ pole in dimensional regularization.

Besides the soft divergence, the integral also diverges when the loop momentum becomes collinear to one of the external momenta. To make this explicit we can close the contour of the l^0 integral in the lower half-plane. The integration over the semi-circle at infinity vanishes since the integrand falls off as $1/(l^0)^3$. Writing the denominators as $l^2 + i0 = (l^0 + |\mathbf{l}| - i0)(l^0 - |\mathbf{l}| + i0)$ we then pick up the residues of the poles below the real axis. Let's look at the residue at $l^0 = |\mathbf{l}| - i0$

$$\begin{aligned} &2\pi \int \frac{d^{d-1}\mathbf{l}}{\pi^{d/2}} \frac{1}{2E_\mathbf{l}} \frac{1}{[(E_\mathbf{l} + E_\mathbf{p})^2 - (\mathbf{l} + \mathbf{p})^2 + i0][(E_\mathbf{l} + E_{\mathbf{p}'})^2 - (\mathbf{l} + \mathbf{p}')^2 + i0]} \\ &= \int \frac{d^{d-1}\mathbf{l}}{\pi^{d/2-1}} \frac{1}{4E_\mathbf{l}} \frac{1}{[E_\mathbf{l} E_\mathbf{p} - \mathbf{l} \cdot \mathbf{p} + i0][E_\mathbf{l} E_{\mathbf{p}'} - \mathbf{l} \cdot \mathbf{p}' + i0]} \end{aligned} \tag{2.11}$$

We observe that the remaining propagators diverge when the loop momentum becomes collinear to the respective external momentum p or p'. In the coordinate system where $\mathbf{p} = E_\mathbf{p}(\vec{0}^\perp, 1)$ and $\mathbf{p}' = E_{\mathbf{p}'}(\sqrt{1-c^2}\vec{n}^\perp_{p'}, c)$ the integral becomes

$$\int \frac{dE_\mathbf{l}}{4E_\mathbf{l}^{5-d}} \int \frac{d\Omega_{d-2}}{\pi^{d/2-1}} \int_{-1}^{1} \frac{d\cos(\vartheta)[1-\cos^2(\vartheta)]^{d/2-2}}{E_\mathbf{p}[1-\cos(\vartheta)]E_{\mathbf{p}'}[1-\sqrt{1-c^2}\sqrt{1-\cos^2(\vartheta)}\vec{n}^\perp_l \cdot \vec{n}^\perp_{p'} - c\cos(\vartheta)]} \tag{2.12}$$

which diverges for $\cos(\vartheta) \to 1$ as $d\cos(\vartheta)/[1-\cos(\vartheta)]^{1+\epsilon}$. Like the situation with the loop momentum becoming soft, we see that the internal propagators go on-shell when the loop momentum becomes collinear to one of the external on-shell momenta. In dimensional regularization *collinear divergences* are regulated by the $\sin^{d-4}(\vartheta)$ factor in the Jacobian and yield $1/\epsilon$ poles. It is clear from the symmetry under $p \leftrightarrow p'$ that an analogous divergence exists when the loop momentum becomes collinear to p'.

Both soft and collinear divergences are classified as *IR divergences*. As we saw they can both be regulated dimensionally. Since the loop momentum can be simultaneously collinear to an external momentum and soft the divergences overlap and produce double poles in ϵ. This can be observed in (2.12) where the energy integration yields a soft pole in addition to the collinear pole in the $\cos(\vartheta)$ integral. In total we find

$$(2.10) = \frac{e^{-\epsilon\gamma_E}}{q^2}\left[\frac{1}{\epsilon^2} - \frac{\ln(-q^2 - i0)}{\epsilon} + \frac{1}{2}\ln^2(-q^2 - i0) - \frac{\pi^2}{12} + \mathcal{O}(\epsilon)\right], \tag{2.13}$$

which will be derived in the next subsection.

Dimensionally regularized loop integrals satisfy the following properties

$$\begin{array}{ll} \text{Linearity:} & \int [dl][af(l) + bg(l)] = a\int [dl]f(l) + b\int [dl]g(l) \\ \text{Scaling:} & \int [dl]f(sl) = s^{-d}\int [dl]f(l) \\ \text{Translation invariance:} & \int [dl]f(l+q) = \int [dl]f(l) \end{array} \tag{2.14}$$

where $a, b \in \mathbb{C}$ and $s \in \mathbb{R}^+$. Another important property is that scaleless integrals vanish. An integral is scaleless when its integrand $f_{\text{scaleless}}(\{l_i\})$ scales homogeneously under rescaling of any of its loop momenta, i.e. when it satisfies the relation

$$f_{\text{scaleless}}(\{l_i\}) \xrightarrow{l_j \to s\, l_j} s^\eta f_{\text{scaleless}}(\{l_i\}) \tag{2.15}$$

for any $j = 1, \ldots, L$. Using the scaling property (2.14) this gives the condition

$$\left[1 - s^{d+\eta}\right]\int [dl]_L f_{\text{scaleless}}(\{l_i\}) = 0 \tag{2.16}$$

valid for all d and we conclude that the integral must vanish. In calculations we therefore always set scaleless integrals to zero directly. To understand how this happens we proceed with the explicit calculation of the simplest case which is the massless tadpole integral with index n

$$\int \frac{d^d l}{i\pi^{d/2}}\frac{1}{[l^2]^n} = (-1)^n \int \frac{d^d l_E}{\pi^{d/2}}\frac{1}{[l_E^2]^n} = (-1)^n \int \frac{d\Omega_d}{\pi^{d/2}}\int_0^\infty d|l_E||l_E|^{d-1-2n} \tag{2.17}$$

The radial integral does not converge for any d since it is UV divergent for $d \geq 2n$ and IR divergent for $d \leq 2n$. We can however split the integral at some arbitrary scale μ

$$\begin{array}{ll} \int_0^\mu d|l_E||l_E|^{d-1-2n} = \frac{\mu^{d-2n}}{d-2n}, & d > 2n \\ \int_\mu^\infty d|l_E||l_E|^{d-1-2n} = -\frac{\mu^{d-2n}}{d-2n}, & d < 2n \end{array} \tag{2.18}$$

The two parts can then be analytically continued to the complex plane after which they add to zero. Alternatively, we can use an additional analytic regulator instead off splitting the integral. We see that for $n = 2$ the two parts contain an IR and UV divergence, respectively, which cancel each other. This is common in dimensional regularization and makes it quite difficult to separate UV and IR divergences. Thus, some additional regularization or cutoff procedure is often required when one wants to determine anomalous dimensions in dimensional regularization.

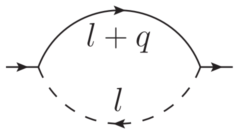


Figure 2: One-loop propagator diagram with one massive (solid) and one massless (dashed) propagator.

2.2 Feynman parametrization

The basic idea of Feynman parametrization is to combine propagators such that the loop integration can be performed in a straightforward way after some momentum shifts and one is left with a parametric integral which can subsequently be solved. Two propagators can be combined with the identity

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{dx x^{a-1}(1-x)^{b-1}}{[xA+(1-x)B]^{a+b}} . \tag{2.19}$$

As an example we consider the massive 1-loop propagator diagram with general indices

$$F(a,b) = \int \frac{[dl]}{[l^2+i0]^a[(l+q)^2-m^2+i0]^b} \tag{2.20}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx x^{a-1}(1-x)^{b-1} \int \frac{[dl]}{[l^2+2(1-x)l\cdot q+(1-x)q^2-xm^2+i0]^{a+b}} \tag{2.21}$$

We can now shift the loop momentum $l \to l-(1-x)q$ to complete the square

$$F(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx x^{a-1}(1-x)^{b-1} \int \frac{[dl]}{[l^2+x(1-x)q^2-(1-x)m^2+i0]^{a+b}} . \tag{2.22}$$

The loop integral can now easily be solved. The angular integration is trivial and given by (2.5) and the radial integral can be performed in terms of the Euler Beta function as in (2.7). The general result is

$$\int \frac{[dl]}{[l^2-\Delta]^\lambda} = (-1)^\lambda \frac{\Gamma(\lambda-d/2)}{\Gamma(\lambda)} \Delta^{d/2-\lambda} . \tag{2.23}$$

We obtain

$$\begin{aligned} F(a,b) =& (-1)^{a+b} \frac{\Gamma(a+b-d/2)}{\Gamma(a)\Gamma(b)} \int_0^1 dx \frac{x^{a-1}(1-x)^{b-1}}{[-x(1-x)q^2+(1-x)m^2-i0]^{a+b-d/2}} \\ =& \frac{(-1)^{a+b}}{(m^2)^{a+b-d/2}} \frac{\Gamma(a+b-d/2)}{\Gamma(a)\Gamma(b)} \int_0^1 dx \frac{x^{a-1}(1-x)^{-a-1+d/2}}{[1-xq^2/m^2-i0]^{a+b-d/2}} \\ =& \frac{(-1)^{a+b}}{(m^2)^{a+b-d/2}} \frac{\Gamma(a+b-d/2)\Gamma(d/2-a)}{\Gamma(b)\Gamma(d/2)} {}_2F_1\left(d/2-a-b, a; d/2; \frac{q^2}{m^2}\right) . \end{aligned} \tag{2.24}$$

Here we have evaluated the Feynman parameter integral in terms of Hypergeometric functions which are defined as

$$
{}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \ldots (\beta_q)_k} \frac{z^k}{k!}, \tag{2.25}
$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. Specifically, we have used the integral representation

$$
{}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dx \frac{x^{\beta-1}(1-x)^{\gamma-\beta-1}}{(1-xz)^{\alpha}} \qquad [\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0]. \tag{2.26}
$$

For specific values of a and b we can then use the `Mathematica` package `HypExp` [2] to expand the result in ϵ. Alternatively, we can expand in ϵ before the x-integral is performed. For instance we have

$$
\begin{aligned}
F(1,1) =& \frac{\Gamma(\epsilon)}{(m^2)^\epsilon} \int_0^1 dx \left\{ 1 - \left[\ln(1-x) + \ln\left(1 - \frac{xq^2}{m^2} - i0\right)\right] \epsilon + \mathcal{O}(\epsilon^2) \right\} \\
=& \frac{\Gamma(\epsilon)}{(m^2)^\epsilon} \left\{ 1 + \left[2 - \left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{q^2}{m^2} - i0\right)\right] \epsilon + \mathcal{O}(\epsilon^2) \right\} \\
=& e^{-\epsilon\gamma_E} \left[\frac{1}{\epsilon} + 2 - \ln(m^2) - \left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{q^2}{m^2} - i0\right) + \mathcal{O}(\epsilon) \right].
\end{aligned} \tag{2.27}
$$

Divergences can however also appear in the Feynman parameter integral. Let's take

$$
F(2,1) = -\Gamma(1+\epsilon) \int_0^1 dx \frac{x}{(1-x)^{1+\epsilon}[m^2 - xq^2 - i0]^{1+\epsilon}} \tag{2.28}
$$

as an example. The integral is divergent at the endpoint $x = 1$ and unlike before we cannot expand the integrand in ϵ because the ϵ in the exponent is needed to regulate the integral. In such cases one can proceed with a subtraction

$$
\begin{aligned}
F(2,1) =& -\Gamma(1+\epsilon) \Bigg\{ \int_0^1 dx \frac{1}{(1-x)^{1+\epsilon}} \left[\frac{x}{[m^2 - xq^2 - i0]^{1+\epsilon}} - \frac{1}{[m^2 - q^2 - i0]^{1+\epsilon}} \right] \\
&+ \frac{1}{[m^2 - q^2 - i0]^{1+\epsilon}} \int_0^1 \frac{dx}{(1-x)^{1+\epsilon}} \Bigg\},
\end{aligned} \tag{2.29}
$$

and the expand the subtracted expression in square brackets in ϵ since the divergence has been extracted into the second term. We find

$$
\begin{aligned}
F(2,1) =& -\Gamma(1+\epsilon) \left\{ \left[\frac{-1}{m^2 - q^2} \int_0^1 dx \frac{1}{1 - \frac{xq^2}{m^2} - i0} + \mathcal{O}(\epsilon) \right] - \frac{1}{\epsilon[m^2 - q^2 - i0]^{1+\epsilon}} \right\} \\
=& \frac{e^{-\epsilon\gamma_E}}{m^2 - q^2} \left[\frac{1}{\epsilon} - \ln(m^2 - q^2 - i0) - \frac{m^2}{q^2} \ln\left(1 - \frac{q^2}{m^2} - i0\right) + \mathcal{O}(\epsilon) \right].
\end{aligned} \tag{2.30}
$$

It is worth noting that the $i0$ prescription becomes relevant for $q^2 \geq m^2$ where the particles in the loop can go on-shell. This is related to the optical theorem which states that the imaginary part of a Feynman diagram is given by the sum over all possible on-shell cuts. The imaginary part in the integrals $F(1,1)$ and $F(2,1)$ considered above is due to the branch cut of the logarithm for negative argument, i.e. with $z > 0$

$$\ln(-z \pm i0) = \ln(z) \pm i\pi. \tag{2.31}$$

If we have more propagators we could apply (2.19) repeatedly until the loop integrals can be performed after completing the square. However, it's often simpler (and less error-prone) to use the more general identity for an arbitrary number of propagators

$$\frac{1}{\prod_{i=1}^N A_i^{\lambda_i}} = \frac{\Gamma(N_\lambda)}{\prod_{i=1}^N \Gamma(\lambda_i)} \int_0^\infty \left[\prod_{i=1}^N dx_i x_i^{\lambda_i - 1}\right] \frac{\delta\left(1 - \sum_{i=1}^N x_i\right)}{\left[\sum_{i=1}^N x_i A_i\right]^{N_\lambda}}, \tag{2.32}$$

where $N_\lambda = \sum_{i=1}^N \lambda_i$. Let us apply this to the massless one-loop vertex integral from Section 2.1

$$\begin{aligned}
(2.10) =& 2\int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) \int \frac{[dl]}{[l^2 + 2l\cdot(x_1 p + x_2 p') + i0]^3} \\
=& 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{[dl]}{[l^2 - 2x_1 x_2 p\cdot p') + i0]^3} \\
=& -\Gamma(1+\epsilon) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{[-x_1 x_2 q^2 - i0]^{1+\epsilon}} \\
=& \frac{\Gamma(1+\epsilon)}{[-q^2 - i0]^{1+\epsilon}} \int_0^1 dx_1 \frac{1}{\epsilon x_1^{1+\epsilon}(1-x_1)^\epsilon} \\
=& \frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\epsilon\Gamma(1-2\epsilon)} \frac{1}{[-q^2 - i0]^{1+\epsilon}} \\
=& -\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \frac{1}{[-q^2 - i0]^{1+\epsilon}}.
\end{aligned} \tag{2.33}$$

Expanding to finite order in ϵ, the result (2.13) is reproduced.

One can also directly write an expression for the result of a Feynman-parametrized integral after the loop integration has been performed. Adopting the conventions of [3, 4] this takes the form

$$\begin{aligned}
F(\{\lambda_i\}) =& \frac{\Gamma(N_\lambda)}{\prod_{i=1}^N \Gamma(\lambda_i)} \int_0^\infty \left[\prod_{i=1}^N dx_i x_i^{\lambda_i - 1}\right] \delta\left(1 - \sum_{i=1}^N x_i\right) \\
& \times \int [dl]_L \left[\sum_{i,j=1}^L M_{ij} l_i \cdot l_j - 2\sum_{i=1}^L Q_i \cdot l_i + J + i0\right]^{-N_\lambda}
\end{aligned} \tag{2.34}$$

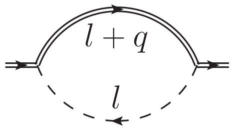


Figure 3: One-loop HQET propagator diagram. The double line indicates a linear propagator with reference vector v.

$$=\frac{(-1)^{N_\lambda}\Gamma\left(N_\lambda-\frac{Ld}{2}\right)}{\prod_{i=1}^{N}\Gamma(\lambda_i)}\int_0^\infty\left[\prod_{i=1}^{N}dx_i x_i^{\lambda_i-1}\right]\delta\left(1-\sum_{i=1}^{N}x_i\right)\frac{\mathcal{U}^{N_\lambda-(L+1)d/2}}{\mathcal{F}^{N_\lambda-Ld/2}}, \quad (2.35)$$

with the so-called graph polynomials

$$\mathcal{F}(\{x_i\}) =\det(M)\left[\sum_{i,j=1}^{L}M_{ij}^{-1}Q_i\cdot Q_j-J-i0\right], \quad (2.36)$$

$$\mathcal{U}(\{x_i\}) =\det(M). \quad (2.37)$$

We note that the graph polynomials can also be constructed from the topology of the corresponding Feynman diagram [1] instead of being obtained from the representation (2.34) by means of (2.36) and (2.37).

Last but not least there is an important generalization of (2.32) and (2.35) related to the alternative Feynman parametrization

$$\frac{1}{A^{\lambda_1}B^{\lambda_2}}=\frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\int_0^\infty dx\frac{x^{\lambda_2-1}}{(A+xB)^{\lambda_1+\lambda_2}}, \quad (2.38)$$

which is especially useful when A is a quadratic and B linear in the loop momentum. This is the Cheng-Wu theorem [5] which states that the delta function in (2.32) and (2.35) can be replaced with

$$\delta\left(1-\sum_{i\in\nu}x_i\right) \quad (2.39)$$

where ν is an arbitrary subset of the propagator labels $1,\ldots,N$ and the other Feynman parameters are thus integrated from 0 to ∞. Clearly, (2.38) is reproduced in the two-propagator case when we only put one of the two Feynman parameters into the delta function.

As an example of dealing with linear propagators we look at the one-loop HQET propagator diagram of Figure 3 where v is a reference vector with $v^2=1$:

$$F_{\mathrm{HQET}}(a,b)=\int\frac{[dl]}{[2v\cdot(l+q)+i0]^a[l^2+i0]^b}$$
$$=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^\infty dx x^{a-1}\int\frac{[dl]}{[l^2+2xv\cdot(l+q)+i0]^{a+b}}$$

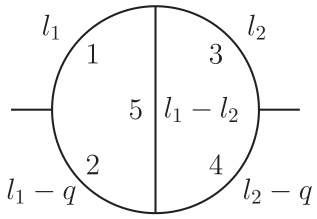


Figure 4: Two-loop propagator diagram.

$$
\begin{aligned}
=&\frac{(-1)^{a+b}\Gamma\left(a+b-d/2\right)}{\Gamma(a)\Gamma(b)}\int\limits_0^\infty \frac{dx x^{a-1}}{\left[x(x-2v\cdot q)+i0\right]^{a+b-d/2}}\\
=&\frac{(-1)^{a+b}\Gamma\left(d/2-b\right)\Gamma\left(a+2b-d\right)}{\Gamma(a)\Gamma(b)}\left(-2v\cdot q+i0\right)^{d-a-2b}\,. \qquad (2.40)
\end{aligned}
$$

Here, the Feynman parameter x has mass dimension one and has been rescaled by $x = (-2v \cdot q + i0)u$ in the last step to perform the integration with (2.8).

The Cheng-Wu theorem is however also useful when there are only quadratic propagators. Consider the 2-loop massless propagator diagram in Figure 4

$$
F_{\obar}(\{a_i\}) = \int \frac{[dl]_2}{[l_1^2]^{a_1}[(l_1-q)^2]^{a_2}[l_2^2]^{a_3}[(l_2-q)^2]^{a_4}[(l_1-l_2)^2]^{a_5}}\,. \qquad (2.41)
$$

The corresponding graph polynomials are

$$
\begin{aligned}
\mathcal{F} =& -q^2[x_{12}x_3x_4 + x_1x_2x_{34} + x_{13}x_{24}x_5] - i0\,, \qquad (2.42)\\
\mathcal{U} =& x_{12}x_{34} + x_{1234}x_5\,, \qquad (2.43)
\end{aligned}
$$

where we have used the notation $x_{ij\ldots k} = x_i + x_j + \cdots + x_k$. Consequently, the integral with $a_i = 1$ takes the form

$$
F_{\obar}(1,1,1,1,1) = -\Gamma\left(1+2\epsilon\right)\int\limits_0^\infty \left[\prod_{i=1}^5 dx_i\right]\delta\left(1-\sum_{i=1}^5 x_i\right)\frac{1}{\mathcal{U}^{1-3\epsilon}\mathcal{F}^{1+2\epsilon}}\,. \qquad (2.44)
$$

The integral can be solved for $\epsilon = 0$ by applying the Cheng-Wu theorem to replace the delta function with $\delta(1 - x_5)$ such that both graph polynomials are linear in each of the remaining Feynman parameters. The remaining integration can then by done in `Mathematica`

$$
\begin{aligned}
F_{\obar}(1,1,1,1,1) =& \frac{1}{q^2}\int\limits_0^\infty \left[\prod_{i=1}^4 dx_i\right]\frac{1}{[x_{12}x_{34}+x_{1234}][x_{12}x_3x_4+x_1x_2x_{34}+x_{13}x_{24}]} + \mathcal{O}(\epsilon)\\
=& \frac{6\zeta(3)}{q^2} + \mathcal{O}(\epsilon)\,. \qquad (2.45)
\end{aligned}
$$

The Riemann Zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}\,, \tag{2.46}$$

with integer arguments often appears in Feynman integrals. At even arguments the Zeta function is proportional to π to the respective power $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, ...

2.3 Other elementary techniques

2.3.1 Alpha representation

An alternative to Feynman parameters is the Alpha representation due to Schwinger. In this approach we write each propagator as an integral over an exponential

$$\frac{i}{P^2 - m^2 + i0} = \int_0^\infty d\alpha e^{i(P^2-m^2+i0)\alpha} \tag{2.47}$$

and then shift the loop momentum to complete the square such that the loop integral takes the form of a Gaussian integral. After this is performed we are left with a α-parameter integral. The Alpha and Feynman parametrizations are related by a change of integration variables and we will therefore not consider Alpha representations further.

2.3.2 Partial fractions

When working with effective theories we often have to perform matching calculations to determine perturbative Wilson coefficients. Since the matching coefficients are independent of the kinematics we can choose a convenient configuration which simplifies the calculation. Let us consider the scalar integral for the one-loop vertex correction in $\gamma^* \to t\bar{t}$ directly at the threshold $q^2 = 4m^2$ where the tops both have momentum $q/2$

$$\begin{aligned} F_{\text{thr}}(n_1, n_2, n_3) &\equiv \int \frac{[dl]}{[(l+q/2)^2 - m^2]^{n_1}[(l-q/2)^2 - m^2]^{n_2}[l^2]^{n_3}} \\ &= \int \frac{[dl]}{[l^2 + l\cdot q]^{n_1}[l^2 - l\cdot q]^{n_2}[l^2]^{n_3}}\,. \end{aligned} \tag{2.48}$$

We observe that the three denominators only contain two scalar products l^2 and $l \cdot q$ that involve the loop momentum. In situations like this we can apply partial fractions to reduce the integral to a sum of integrals with a smaller number of different propagators which are easier to solve

$$\begin{aligned} F_{\text{thr}}(1,1,1) &= \int \frac{[dl]}{2[l^2]^2}\left(\frac{1}{l^2 + l\cdot q} + \frac{1}{l^2 - l\cdot q}\right) \\ &= \left[F_{\text{thr}}(1,0,2) + F_{\text{thr}}(0,1,2)\right]/2\,. \end{aligned} \tag{2.49}$$

The formula for arbitrary integer powers n_1, n_2 of the denominators is given by

$$\frac{1}{[a+b]^{n_1}[a+c]^{n_2}} = \sum_{i=0}^{n_1-1} \binom{i+n_2-1}{n_2-1} \frac{(-1)^i}{[c-b]^{n_2+i}[a+b]^{n_1-i}} + \sum_{i=0}^{n_2-1} \binom{i+n_1-1}{n_1-1} \frac{(-1)^{n_1}}{[c-b]^{n_1+i}[a+c]^{n_2-i}} . \tag{2.50}$$

Since more complicated problems are solved on computers anyways one can however proceed algorithmically instead of using more general partial fractions identities. E.g. in `Mathematica` we could use

```
Fthr[5,8,3] //. {Fthr[n1_,n2_,n3_] /; (n1 > 0 && n2 > 0)
        →(Fthr[n1,n2-1,n3+1]+Fthr[n1-1,n2,n3+1])/2} // Expand
```

2.4 (Avoiding) tensor integrals

In practical calculations we do not only encounter scalar integrals (1.1) but also tensors

$$F^{\mu_{11}\cdots\mu_{LM}}(\{n_i\}) = \int [dl]_L \frac{(l_1^{\mu_{11}} \dots l_1^{\mu_{1K}}) \dots (l_L^{\mu_{L1}} \dots l_L^{\mu_{LM}})}{[P_1^2 - m_1^2 + i0]^{n_1} \dots [P_N^2 - m_N^2 + i0]^{n_N}} , \tag{2.51}$$

To solve simple integrals we can proceed with Feynman parameters and exploit the symmetries of the integral to simplify the numerator. Consider the example

$$\begin{aligned}
F^{\mu\nu\rho} &= \int [dl] \frac{l^\mu l^\nu l^\rho}{l^2 (l+q)^2} \\
&= \int_0^1 dx \int [dl] \frac{l^\mu l^\nu l^\rho}{[l^2 + 2xl\cdot q + xq^2]^2} \\
&= \int_0^1 dx \int [dl] \frac{(l-xq)^\mu (l-xq)^\nu (l-xq)^\rho}{[l^2 + x(1-x)q^2]^2} \\
&= \int_0^1 dx \int [dl] \frac{l^\mu l^\nu l^\rho - x(l^\mu l^\nu q^\rho + l^\mu q^\nu l^\rho + q^\mu l^\nu l^\rho) + x^2 (l^\mu q^\nu q^\rho + \dots) - x^3 q^\mu q^\nu q^\rho}{[l^2 + x(1-x)q^2]^2} \\
&= -\int_0^1 dx x \int [dl] \frac{\frac{l^2}{d}(g^{\mu\nu} q^\rho + g^{\mu\rho} q^\nu + q^\mu g^{\nu\rho}) + x^2 q^\mu q^\nu q^\rho}{[l^2 + x(1-x)q^2]^2} \\
&= \int_0^1 dx x \left[\frac{\Gamma(-1+\epsilon)}{2} \frac{g^{\mu\nu} q^\rho + g^{\mu\rho} q^\nu + q^\mu g^{\nu\rho}}{[-x(1-x)q^2 - i0]^{-1+\epsilon}} - x^2 \Gamma(\epsilon) \frac{q^\mu q^\nu q^\rho}{[-x(1-x)q^2 - i0]^\epsilon} \right] \\
&= -(-q^2 - i0)^{-\epsilon} \left[\frac{\Gamma(-1+\epsilon)\Gamma(2-\epsilon)\Gamma(3-\epsilon)}{2\,\Gamma(5-2\epsilon)} q^2 (g^{\mu\nu} q^\rho + g^{\mu\rho} q^\nu + q^\mu g^{\nu\rho}) \right. \\
&\qquad \left. + \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(4-\epsilon)}{\Gamma(5-2\epsilon)} q^\mu q^\nu q^\rho \right]
\end{aligned} \tag{2.52}$$

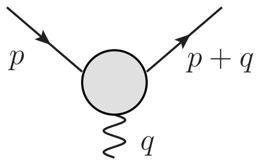


Figure 5: QED vertex.

In the calculation we have shifted the loop momentum by $l \to l - xq$ to complete the square, i.e. turning the denominator into a function of l^2 with no angular dependence. In general, we have

$$\int [dl] \frac{l^{\mu_1} \dots l^{\mu_n}}{f(l^2)} = \begin{cases} \frac{g^{\mu_1\mu_2}\dots g^{\mu_{n-1}\mu_n} + \text{permutations}}{d(d+2)\dots(d+n-2)} \int [dl] \frac{(l^2)^{n/2}}{f(l^2)}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{2.53}$$

By anticipating the structure of the result (2.52)

$$F^{\mu\nu\rho} = A(q^2)q^2(g^{\mu\nu}q^\rho + g^{\mu\rho}q^\nu + q^\mu g^{\nu\rho}) + B(q^2)q^\mu q^\nu q^\rho \tag{2.54}$$

from symmetry arguments we can also avoid the evaluation of tensor integrals altogether and solve the system of equations

$$g_{\mu\nu}q_\rho F^{\mu\nu\rho} = [A(q^2)(d+2) + B(q^2)](q^2)^2 \tag{2.55}$$
$$q_\mu q_\nu q_\rho F^{\mu\nu\rho} = [3A(q^2) + B(q^2)](q^2)^3 \tag{2.56}$$

for the functions $A(q^2)$ and $B(q^2)$ in terms of the scalar integrals on the left-hand side. This is the spirit of the Passarino-Veltman tensor reduction technique [6].

In practice it is often preferable to avoid tensor integrals by projecting amplitudes onto scalar form factors. As an example we consider the QED vertex given by Figure 5

$$i\mathcal{A}^\mu = ie\,\bar{u}(p+q)\Gamma^\mu(p,q)u(p). \tag{2.57}$$

From the on-shell conditions $p^2 = (p+q)^2 = m^2$ we derive $p \cdot q = -q^2/2$. Thus there is only one kinematic invariant q^2. This allows us to decompose the vertex function into form factors that only depend on q^2 (we suppress dependence on m^2 in the notation)

$$\Gamma^\mu(p,q) = A(q^2)\gamma^\mu + B(q^2)p^\mu/m + C(q^2)q^\mu/m\,. \tag{2.58}$$

We do not allow terms involving γ^5 which would violate parity. From the Ward identity $q_\mu \mathcal{A}^\mu = 0$ we obtain the constraint $C(q^2) = B(q^2)/2$. Application of the Gordon identity transfers this to the standard form

$$\Gamma^\mu(p,q) = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}, \tag{2.59}$$

with $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$, which involves the two form factors $F_{1,2}(q^2)$. Now we can construct projectors onto these form factors from the ansatz

$$F_1(q^2) = \mathrm{Tr}\left[(\not{p} + \not{q} + m)\Gamma^\mu(p,q)(\not{p} + m)\left(a_1\gamma_\mu + b_1\frac{(2p+q)_\mu}{2m}\right)\right], \tag{2.60}$$

$$F_2(q^2) = \mathrm{Tr}\left[(\not{p} + \not{q} + m)\Gamma^\mu(p,q)(\not{p} + m)\left(a_2\gamma_\mu + b_2\frac{(2p+q)_\mu}{2m}\right)\right], \tag{2.61}$$

which follows from closing the fermion loop by inserting a vertex with coefficients that can be determined by inserting (2.59) and solving the resulting system of equations. Here, we have used the structure (2.58) instead of (2.59) in the ansatz to reduce the number of Dirac matrices which appear in the traces. These projectors then solve the problem of dealing with tensor integrals at arbitrary loop orders and have indeed been used for the corresponding three-loop calculations in the planar (large-N_c) limit [7, 8].

2.5 Exercises

1. Solve the 2-loop vacuum diagram with one or two massive lines

$$F_{\text{vac1}}(a_1, a_2, a_3) = \int \frac{[dl]_2}{[l_1^2 + i0]^{a_1}[l_2^2 + i0]^{a_2}[(l_1 - l_2)^2 - m^2 + i0]^{a_3}}, \tag{2.62}$$

$$F_{\text{vac2}}(a_1, a_2, a_3) = \int \frac{[dl]_2}{[l_1^2 - m^2 + i0]^{a_1}[l_2^2 - m^2 + i0]^{a_2}[(l_1 - l_2)^2 + i0]^{a_3}}, \tag{2.63}$$

   for general indices using Feynman parametrization.

2. Reproduce the result (2.52) using the idea of Passsarino-Veltman tensor reduction.

3. Determine the projectors (2.60) and (2.61). Apply them to reproduce Schwinger's seminal 1948 result $(g-2)_e = \alpha/\pi$ from the relation

$$g = 2\left[F_1(0) + F_2(0)\right] = 2 + 2F_2(0), \tag{2.64}$$

   where the latter identity is due to the on-shell renormalization condition $F_1(0) = 1$. Hint: A convenient way to do this is to use the `Mathematica` package `Package-X` which evaluates traces as well as one-loop integrals. See `https://packagex.hepforge.org/` for the downloads and documentation.

Files: `Examples/General_Feynman_Parametrization.m`, `Exercises/Basics_gMinus2.m`

3 Sector decomposition

The analytic integration of more complicated Feynman parameter integrals can be extremely difficult. In such cases it would be welcome to be able to integrate numerically. This is however not straightforward in dimensional regularization. For instance, we already observed in the example (2.28) that the Feynman parameter integrals can contain divergences. Since one has to expand in the dimensional regulator ϵ before performing a numerical integration, the regularization of such divergences is spoiled if this is done naively. In the considered case the solution was straightforward, but in general we have IR divergences in multiple kinematic configurations which are overlapping. A solution to this problem is sector decomposition where the original Feynman parameter integral is decomposed into sectors where the overlapping singularities are disentangled and the integrations can be performed numerically after the remaining singularities which only exist in factorized form are subtracted. Obviously, the preferred strategy to achieve this would be the one that ends up with the smallest possible number of sectors. There are by now many different algorithmic strategies. Here, we will review the original recursive sector decomposition from [3] to illustrate the basic idea. A number of sector decomposition strategies are implemented in the public codes `FIESTA` [9] and `pySecDec` [10].

For convenience we strip the Feynman parameter integral (2.35) of its prefactors and define

$$\hat{F}(\{\lambda_i\}) = \int_0^\infty \left[\prod_{i=1}^N dx_i x_i^{\lambda_i - 1}\right] \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{\mathcal{U}^{N_\lambda - (L+1)d/2}}{\mathcal{F}^{N_\lambda - Ld/2}} . \tag{3.1}$$

Let us list some properties of this representation which will be relevant here:

1. An overall UV divergence manifests as a pole of the Gamma function $\Gamma(N_\lambda - Ld/2)$ multiplying (3.1).

2. $\mathcal{U}$ is a positive semi-definite function in the integration domain where all $x_i \geq 0$. A necessary condition for the existence of UV subdivergences is that $\mathcal{U}$ vanishes.

3. In the Euclidean region (all Mandelstam invariants are negative) $\mathcal{F}$ is a positive semi-definite function in the domain where all $x_i \geq 0$. A necessary condition for the existence of IR divergences are the Landau equations [11] and in particular that $\mathcal{F}$ vanishes.

4. $\mathcal{U}$ is a homogeneous polynomial of the Feynman parameters of degree L, i.e. all terms have in total L powers of Feynman parameters.

5. $\mathcal{F}$ is a homogeneous polynomial of the Feynman parameters of degree $L+1$ with coefficients depending on the kinematic invariants of the corresponding diagram.

In the following we will assume that $\mathcal{F}$ is either positive or negative semi-definite as in the case of an Euclidean momentum configuration, see item 3. According to items 2–5 divergences can then only occur for configurations where some subset of the Feynman parameters vanishes. In general, singularities can also appear inside the integration region instead of at the endpoint, e.g. when thresholds are crossed. This case can be treated with contour deformations of the Feynman parameters into the complex plane [12, 13] which we do not discuss here.

The first step of the algorithm differs from the following recursive procedure due to the presence of the delta function. We divide (3.1) into N primary sectors where one Feynman parameter is larger than all others by multiplying with

$$\sum_{n=1}^{N} \prod_{m \neq n} \theta(x_n - x_m)\,. \tag{3.2}$$

The contribution from the nth term in the sum is called the nth *primary sector* $\hat{F}_n$. We then substitute

$$x_i \to \begin{cases} x_n t_i, & i \neq n, \\ x_n, & i = n, \end{cases} \tag{3.3}$$

which allows us to factor out the Feynman parameter x_n from the graph polynomials due to their homogeneity 4 and 5 and to eliminate the delta function by performing the integration over x_n

$$\begin{aligned} \hat{F}_n(\{\lambda_i\}) &= \int\limits_0^\infty dx_n x_n^{N_\lambda - 1} \int_0^1 \left[\prod_{i\neq n} dt_i t_i^{\lambda_i - 1}\right] \delta\left(1 - x_n\left(1 + \sum_{i\neq n} t_i\right)\right) \\ &\quad \times \frac{\left[x_n^L\, \mathcal{U}_n(\{t_i\})\right]^{N_\lambda - (L+1)d/2}}{\left[x_n^{L+1}\, \mathcal{F}_n(\{t_i\})\right]^{N_\lambda - Ld/2}} \\ &= \int_0^1 \left[\prod_{i\neq n} dt_i t_i^{\lambda_i - 1}\right] \frac{\mathcal{U}_n(\{t_i\})^{N_\lambda - (L+1)d/2}}{\mathcal{F}_n(\{t_i\})^{N_\lambda - Ld/2}}\,, \end{aligned} \tag{3.4}$$

where $\mathcal{U}_n$ and $\mathcal{F}_n$ follow from applying the substitution (3.3) to $\mathcal{U}$ and $\mathcal{F}$ and dividing by x_n^L and x_n^{L+1}, respectively. Note that any singularities in (3.4) are due to subsets of $\{t_i\}$ vanishing.

Now we apply a recursive decomposition to integrals of the kind

$$\hat{F}_{n_1 n_2 \ldots}(\{\lambda_i\}) = \int_0^1 \left[\prod_{i\neq n_1} dt_i t_i^{A_i - B_i \epsilon}\right] \frac{\mathcal{U}_{n_1 n_2 \ldots}(\{t_i\})^{N_\lambda - (L+1)d/2}}{\mathcal{F}_{n_1 n_2 \ldots}(\{t_i\})^{N_\lambda - Ld/2}}\,, \tag{3.5}$$

which obviously include the primary sectors (3.4). Given such an integral we determine a minimal set $\mathcal{S} = t_{a_1}, \ldots, t_{a_r}$ such that $\mathcal{U}_{n_1 n_2 \ldots}$ or $\mathcal{F}_{n_1 n_2 \ldots}$ vanish when the elements of $\mathcal{S}$ are set to zero. Note that we only have to do this when the $\mathcal{U}$ and $\mathcal{F}$ polynomials generate divergences, i.e. when their exponents are negative as ϵ goes to zero. For diagrams which do not exhibit UV subdivergences it is sufficient to apply the decomposition exclusively to the $\mathcal{F}$ polynomial. Next we split the r-cube spanned by the elements of $\mathcal{S}$ into r

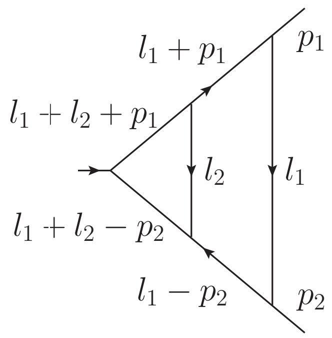


Figure 6: Two-loop planar vertex.

subsectors

$$\sum_{i=1}^{r} \prod_{j \neq i} \theta(t_{a_i} - t_{a_j}) \,, \tag{3.6}$$

and then apply the substitution

$$t_i \to \begin{cases} t_j t_i, & i \neq j, \\ t_j, & i = j, \end{cases} \tag{3.7}$$

to each of the subsectors $n_1 n_2 \dots j$, which transforms the resulting expressions back into the form (3.5). What we have achieved is that at each recursion step a factor t_j has factorized from the $\mathcal{U}$ and/or $\mathcal{F}$ polynomial. This is to be repeated until no more set $\mathcal{S}$ can be found at which point all the singularities in the sectors (3.5) are factorized as $t_i^{A_i - B_i \epsilon}$ with $A_i < 0$.

Now that the singularities are made explicit, they can be subtracted to make the sectors suitable for numerical integration. For each sector we go through all integration variables t_i with respect to which we encounter expressions of the form

$$I_i = \int_0^1 t_i^{A_i - B_i \epsilon} f_i(t_i) \,. \tag{3.8}$$

If $A_i \geq 0$ nothing has to be done. If $A_i < 0$ we subtract the singularity by splitting the integral as follows

$$I_i = \sum_{j=0}^{|A_i|-1} \frac{f_i^{(j)}}{A_i + j + 1 - B_i \epsilon} + \int_0^1 t_i^{A_i - B_i \epsilon} \left[f_i(t_i) - \sum_{j=0}^{|A_i|-1} f_i^{(j)} t_i^j \right] \,, \tag{3.9}$$

where $f_i^{(j)}$ are the Taylor coefficients of the expansion of f_i around $t_i = 0$. The singularities are absorbed into the first terms of (3.9). After this has been done for all t_i the remaining integrals are finite and can be expanded in ϵ without causing unregularized divergences. Last but not least the integrals have to be performed which is typically done numerically. The diagram shown in Figure 6 and some simple one-loop examples are considered in a `Mathematica` file.

Files: `Examples/SectorDecomposition.m`

4 Mellin-Barnes representations

The basic idea of using Mellin Barnes techniques in Feynman integrals is to simplify the integrand, in particular denominators, at the cost of introducing a Mellin Barnes (MB) integral. After the order of integration is interchanged the Feynman parameter integral can then be performed leaving the MB integral for subsequent evaluation. The Mellin Barnes representation for denominators with a sum over two terms is

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}\,, \tag{4.1}$$

where the contour must be chosen such that all the poles stemming from Gamma functions $\Gamma(a+bz)$ with $b>0$ are to the left of the contour and all the poles stemming from Gamma functions $\Gamma(a-bz)$ with $b>0$ are to the right of the contour. For convenience we simply call them left and right poles, respectively. A possible contour is shown in Figure 7. Now, we can check (4.1). The Gamma function falls off exponentially when

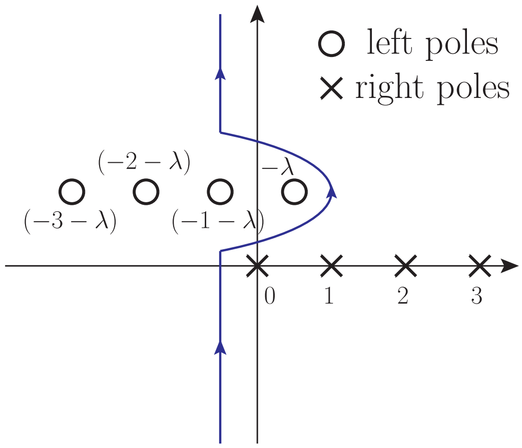


Figure 7: Pole structure of the integrand in (4.1) and possible integration contour.

the argument goes to $\pm i\infty$ with the following behaviour

$$\Gamma(a\pm ib) \simeq \sqrt{2\pi}\, e^{\pm i\frac{\pi}{4}(2a-1)}\, e^{\pm ib(\log b-1)}\, e^{-\frac{b\pi}{2}}\, b^{a-1/2}, \tag{4.2}$$

for $a,b\in\mathbb{R}$ and $b\gg 0$. Thus, we can close the contour to the right and obtain a sum over all the residues of the right poles with a minus sign since the contour encloses the poles in the mathematically negative way

$$\begin{aligned}
&-\frac{1}{\Gamma(\lambda)X^\lambda}\sum_{n=0}^{\infty}\mathrm{Res}_{z=n}\Gamma(\lambda+z)\Gamma(-z)\left(\frac{Y}{X}\right)^z \\
&=-\frac{1}{\Gamma(\lambda)X^\lambda}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\Gamma(\lambda+n)}{n!}\left(\frac{Y}{X}\right)^n
\end{aligned}$$

$$
= \frac{1}{X^\lambda}\left(1+\frac{Y}{X}\right)^{-\lambda} = \frac{1}{(X+Y)^\lambda}\,. \tag{4.3}
$$

Of course the same result is obtained when the contour is closed to the left. The representation (4.1) can either be used to convert a massive propagator into a massless one which is useful when the corresponding massless integral is known, or to simplify a Feynman parameter integral such that it can be performed in terms of Gamma functions. This involves exchanging the order of the MB integral with the loop or Feynman integrals. In this step it is crucial to respect the contour prescription given above also with regards to Gamma functions that are created by the loop or Feynman parameter integrals. This will become clear once we consider examples below.

The last step is then to evaluate the remaining MB integrals. In some cases they can be solved for general ϵ in terms of Gamma functions with the first and second Barnes lemma

$$
\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z) = \frac{\Gamma(\lambda_{13})\Gamma(\lambda_{14})\Gamma(\lambda_{23})\Gamma(\lambda_{24})}{\Gamma(\lambda_{1234})}\,, \tag{4.4}
$$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_{12345}+z)} \\
=&\frac{\Gamma(\lambda_{14})\Gamma(\lambda_{24})\Gamma(\lambda_{34})\Gamma(\lambda_{15})\Gamma(\lambda_{25})\Gamma(\lambda_{35})}{\Gamma(\lambda_{1245})\Gamma(\lambda_{1345})\Gamma(\lambda_{2345})}\,,
\end{aligned} \tag{4.5}
$$

where $\lambda_{13} = \lambda_1 + \lambda_3$ and so forth, or in terms of the hypergeometric function which has the following Mellin Barnes representation

$$
{}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(c+z)}(-x)^z\,. \tag{4.6}
$$

A list with many additional MB integrals is given in the appendices of [1]. Alternatively, one can close the contours to sum up an infinite series of poles as we did in (4.3). Typically this however requires us to first resolve the singularities in ϵ which we will discuss based on examples below. In cases where the analytic expressions for the infinite sums are not known the MB integrals can be evaluated numerically after the singularities have been resolved.

The generalization of (4.1) follows from repeated application

$$
\begin{aligned}
\frac{1}{(X_1+\cdots+X_n)^\lambda} =& \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{+i\infty} dz_2\ldots\int_{-i\infty}^{+i\infty} dz_n\prod_{i=2}^{n} X_i^{z_i} \\
&\times X_1^{-\lambda-z_2-\cdots-z_n}\Gamma(\lambda+z_2+\cdots+z_n)\prod_{i_2}^{n}\Gamma(-z_i)\,.
\end{aligned} \tag{4.7}
$$

Let us now consider the massless box diagram in Figure 8 as a simple example. We have

$$
F_\square(a_1,a_2,a_3,a_4) = \int\frac{[dl]}{[l^2]^{a_1}[(l+p_1)^2]^{a_2}[(l+p_1+p_2)^2]^{a_3}[(l+p_3)^2]^{a_4}}\,, \tag{4.8}
$$

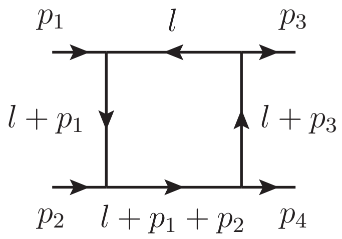


Figure 8: One-loop box.

with the on-shell conditions $p_1^2 = p_2^2 = p_3^2 = (p_1+p_2-p_3)^2 = 0$ and the kinematic invariants $s = (p_1+p_2)^2$ and $t = (p_1-p_3)^2$. The general Feynman parametrization (2.35) takes the form

$$\begin{aligned} F_\Box(a_1,a_2,a_3,a_4) =& \frac{(-1)^{a_{1234}}\Gamma(a_{1234}-2+\epsilon)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} \int_0^\infty \left[\prod_{i=1}^4 dx_i x_i^{a_i-1}\right] \delta(1-x_{1234}) \\ & \times (x_{1234})^{a_{1234}-4+2\epsilon}(-sx_1x_3 - tx_2x_4)^{2-a_{1234}-\epsilon}\,, \end{aligned} \tag{4.9}$$

where $x_{1234} = x_1+x_2+x_3+x_4$ etc. We change variables to $x_1 = u_1v_1$, $x_2 = u_1(1-v_1)$, $x_3 = u_2v_2$ and $x_4 = u_2(1-v_2)$ which gives the Jacobian u_1u_2

$$\begin{aligned} F_\Box(a_1,a_2,a_3,a_4) =& \frac{(-1)^{a_{1234}}\Gamma(a_{1234}-2+\epsilon)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} \int_0^\infty du_1du_2 \int_0^1 dv_1dv_2\delta(1-u_1-u_2) \\ & \times \frac{u_1^{a_{12}-1}u_2^{a_{34}-1}v_1^{a_1-1}(1-v_1)^{a_2-1}v_2^{a_3-1}(1-v_2)^{a_4-1}}{[u_1u_2(-sv_1v_2 - t(1-v_1)(1-v_2))]^{a_{1234}-2+\epsilon}}\,. \end{aligned} \tag{4.10}$$

We use the delta function to eliminate the u_2 integration and obtain after performing the u_1 integral

$$\begin{aligned} F_\Box(a_1,a_2,a_3,a_4) =& \frac{(-1)^{a_{1234}}\Gamma(a_{1234}-2+\epsilon)\Gamma(2-a_{12}-\epsilon)\Gamma(2-a_{34}-\epsilon)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(4-a_{1234}-2\epsilon)} \\ & \times \int_0^1 dv_1dv_2 \frac{v_1^{a_1-1}(1-v_1)^{a_2-1}v_2^{a_3-1}(1-v_2)^{a_4-1}}{[-sv_1v_2 - t(1-v_1)(1-v_2)]^{a_{1234}-2+\epsilon}}\,. \end{aligned} \tag{4.11}$$

Now, we see that the remaining integrals can be performed in terms of Gamma functions when we apply (4.1) to the denominator

$$\begin{aligned} F_\Box(a_1,a_2,a_3,a_4) =& \frac{(-1)^{a_{1234}}\Gamma(2-a_{12}-\epsilon)\Gamma(2-a_{34}-\epsilon)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(4-a_{1234}-2\epsilon)(-s)^{a_{1234}-2+\epsilon}} \\ & \times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(a_{1234}+z-2+\epsilon)\Gamma(-z)\left(\frac{t}{s}\right)^z \\ & \times \int_0^1 dv_1dv_2 v_1^{1-\epsilon-a_{234}-z}(1-v_1)^{a_2-1+z}v_2^{1-\epsilon-a_{124}-z}(1-v_2)^{a_4-1+z} \end{aligned}$$

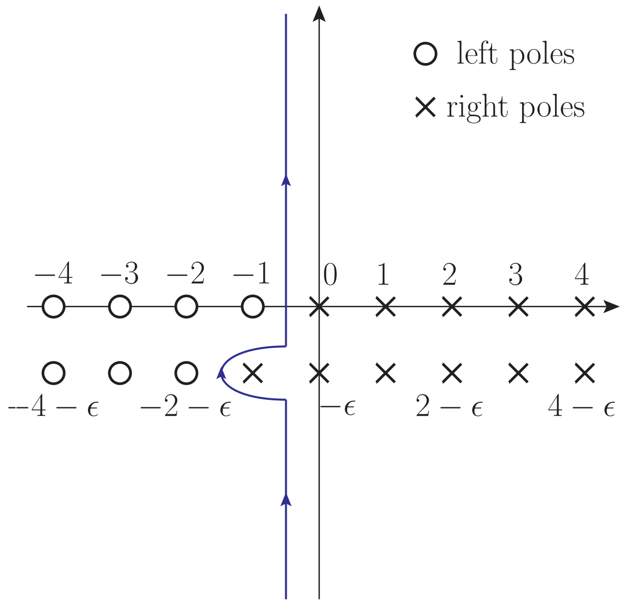


Figure 9: Pole structure of the integrand in (4.13) and integration contour for $\epsilon = i$.

$$
\begin{aligned}
=&\frac{(-1)^{a_{1234}}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(4-a_{1234}-2\epsilon)(-s)^{a_{1234}-2+\epsilon}} \\
&\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \left(\frac{t}{s}\right)^z \Gamma(a_2+z)\Gamma(a_4+z)\Gamma(a_{1234}+z-2+\epsilon) \\
&\times \Gamma(-z)\Gamma(2-\epsilon-a_{234}-z)\Gamma(2-\epsilon-a_{124}-z)\,.
\end{aligned} \tag{4.12}
$$

In particular, we have

$$
F_\square\left(\vec{1}\right) = \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \left(\frac{t}{s}\right)^z \Gamma^2(1+z)\Gamma(2+\epsilon+z)\Gamma(-z)\Gamma^2(-1-\epsilon-z)\,, \tag{4.13}
$$

where the contour must be chosen such that it is to the left of all the poles originating from $\Gamma(-z)\Gamma^2(-1-\epsilon-z)$ and to the right of all the poles from $\Gamma^2(1+z)\Gamma(2+\epsilon+z)$. We note that we are not allowed at this stage to perform manipulations of the integrand like $\Gamma(1+z)\Gamma(-z) \to -\Gamma(z)\Gamma(1-z)$ as this changes the left/right description of some of the poles. A possible contour choice is indicated in Figure 9. We see that the poles at -1 and $-1-\epsilon$ are of opposite nature and thus pinch the integration contour, which must pass in between, as $\epsilon \to 0$. We can deform the integration contour across the pole at $-1-\epsilon$ and then transform it into a parallel to the imaginary axis at $\mathrm{Re}(z) = 1/2$. Since we have crossed the pole we have to take into account its residue. Overall we have

$$
F_\square\left(\vec{1}\right) = \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)}\Bigg[- \mathrm{Res}_{z=-1-\epsilon}\left(\frac{t}{s}\right)^z \Gamma^2(1+z)\Gamma(2+\epsilon+z)\Gamma(-z)\Gamma^2(-1-\epsilon-z)
$$

$$+\frac{1}{2\pi i}\int_{\mathrm{Re}(z)=-\frac{1}{2}} dz\left(\frac{t}{s}\right)^z \Gamma^2(1+z)\Gamma(2+\epsilon+z)\Gamma(-z)\Gamma^2(-1-\epsilon-z)\Bigg], \tag{4.14}$$

where we can now expand the integrand in ϵ since there is no more pinching when $\epsilon \to 0$. We observe that the residue in (4.14) gives a (double) pole in ϵ. This is typical when we shift the contour over poles that pinch it and this procedure is therefore also called resolving the singularities in a MB integral. Once all the singularities have been resolved and there is no more pinching we can fix the remaining integration contours and then perform any manipulations of the integrand because the left/right nature of all poles has been fixed with the contour.

With $x = t/s$ we obtain for the integral in (4.14)

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathrm{Re}(z)=-\frac{1}{2}} dz x^z \Gamma^2(1+z)\Gamma(2+z)\Gamma(-z)\Gamma^2(-1-z) + \mathcal{O}(\epsilon)\\
=&-\sum_{n=0}^{\infty}\mathrm{Res}_{z=n}\left[x^z\Gamma^2(1+z)\Gamma(2+z)\Gamma(-z)\Gamma^2(-1-z)\right] + \mathcal{O}(\epsilon)\\
=&\sum_{n=0}^{\infty}(-x)^n\left[\frac{1}{(n+1)^3}-\frac{\log(x)}{(n+1)^2}+\frac{\log^2(x)+\pi^2}{2(n+1)}\right]+\mathcal{O}(\epsilon)\\
=&\frac{\pi^2\log(x+1)+\log(x+1)\log^2(x)+2\log(x)\mathrm{Li}_2(-x)-2\mathrm{Li}_3(-x)}{2x}\,.
\end{aligned} \tag{4.15}$$

The polylogarithm Li_s has the power-series definition

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}\,, \tag{4.16}$$

inside its radius of convergence $|z| < 1$ and can be analyically continued to the complex plane where it exhibits a branch cut at $\mathrm{Re}(z) \geq 1$. The special case $s = 1$ gives $\mathrm{Li}_1(z) = -\ln(1-z)$. To obtain the complete result we have to add the residue in (4.14) which yields

$$\begin{aligned}
F_{\Box}\left(\vec{1}\right) =&\frac{(-s)^{-\epsilon}e^{-\epsilon\gamma_E}}{st}\Bigg[\frac{4}{\epsilon^2}-\frac{2\log(x)}{\epsilon}-\frac{4\pi^2}{3}+\Bigg(\frac{7\pi^2}{6}\log(x)+\frac{1}{3}\log^3(x)-\frac{34}{3}\zeta(3)\\
&-\left(\log^2(x)+\pi^2\right)\log(1+x)-2\log(x)\mathrm{Li}_2(-x)+2\mathrm{Li}_3(-x)\Bigg)\epsilon+\mathcal{O}(\epsilon^2)\Bigg]\,.
\end{aligned} \tag{4.17}$$

Another useful application of MB representations is the possibility to obtain expansions in some parameters. Let us assume we want to evaluate (4.13) for $t \ll s$. Then we can simply close the contour to the right and pick up the residues of the first few right poles (with a minus) to obtain the first few terms in the expansion. On the other hand the residues of the first few poles on the left give the result as an expansion for $s \ll t$. We find that the leading term for $t \ll s$ scales as $t^{-1-\epsilon}$ due to the pole at $z = -1 - \epsilon$, i.e. it is singular for $t \to 0$ and cannot be obtained by setting $t = 0$ or expanding in t in the Feynman parameter integral. Together with the expansion by regions discussed in Section 7 MB techniques thus provide a method to obtain expansions in such situations.

Resolving the singularities can be challenging when there are multiple MB integrations. We briefly explain the approach called Strategy B in [1] with the example of the massless one-loop box with one off-shell leg $p_1^2 = M^2$. A two-dimensional MB representation is derived in [1] and reads

$$\begin{aligned} F_{\mathrm{OSB}}\left(\vec{1}\right) =& \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_2 \int_{-i\infty}^{+i\infty} dz_4 \frac{(-M^2)^{z_2}(-t)^{z_4}}{(-s)^{z_{24}}} \Gamma(2+\epsilon+z_{24}) \\ & \times \Gamma(1+z_{24})\Gamma(1+z_4)\Gamma(-1-\epsilon-z_4)\Gamma(-1-\epsilon-z_{24})\Gamma(-z_2)\Gamma(-z_4)\,. \end{aligned} \tag{4.18}$$

In a first step we choose a value of ϵ that allows us to fix the contours as straight lines parallel to the imaginary axis. This is possible when the arguments of all Gamma functions are positive as $z_i \to 0$, or more generally when only either left or right poles have zero argument. Here, the choice $\epsilon = -1$ provides such an option with possible contours having $-1 < \mathrm{Re}(z_2) < 0$, $-1 < \mathrm{Re}(z_4) < 0$ and $-1 < \mathrm{Re}(z_2 + z_4) < 0$. For definiteness let us choose the contours with $\mathrm{Re}(z_2) = -1/2$ and $\mathrm{Re}(z_4) = -1/4$. In general such a value of ϵ does not always exist in which case one has to introduce additional analytic regulators.

Now, we take $\epsilon \to 0$ while keeping the contours fixed and account for the residues of poles that cross the contours along the way. For illustration let us change the variables to $\epsilon = -1 + \tilde{\epsilon}$, $z_2 = -1/2 + \tilde{z}_2$ and $z_4 = -1/4 + \tilde{z}_4$ in the integrand

$$F_{\mathrm{OSB}}^{(\tilde{\epsilon}<\frac{1}{4})}\left(\vec{1}\right) = \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)} \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(\tilde{z}_2)=0} d\tilde{z}_2 \int_{\mathrm{Re}(\tilde{z}_4)=0} d\tilde{z}_4 \, f_{\mathrm{OSB}}(\tilde{\epsilon}, \tilde{z}_2, \tilde{z}_4)\,, \tag{4.19}$$

$$\begin{aligned} f_{\mathrm{OSB}}(\tilde{\epsilon}, \tilde{z}_2, \tilde{z}_4) =& \frac{(-M^2)^{-\frac{1}{2}+\tilde{z}_2}(-t)^{-\frac{1}{4}+\tilde{z}_4}}{(-s)^{-\frac{3}{4}+\tilde{z}_{24}}} \Gamma\left(\frac{1}{4}+\tilde{\epsilon}+\tilde{z}_{24}\right)\Gamma\left(\frac{1}{4}+\tilde{z}_{24}\right)\Gamma\left(\frac{3}{4}+\tilde{z}_4\right) \\ & \times \Gamma\left(\frac{1}{4}-\tilde{\epsilon}-\tilde{z}_4\right)\Gamma\left(\frac{3}{4}-\tilde{\epsilon}-\tilde{z}_{24}\right)\Gamma\left(\frac{1}{2}-\tilde{z}_2\right)\Gamma\left(\frac{1}{4}-\tilde{z}_4\right)\,, \end{aligned} \tag{4.20}$$

where the $\tilde{z}_i$ contours coincide with the imaginary axis and we have indicated that this expression is valid for $\tilde{\epsilon} < 1/4$. Now this expression must be analytically continued for $\tilde{\epsilon} \to 1$. As $\tilde{\epsilon}$ crosses $1/4$ the 'left-most' pole of $\Gamma(1/4 - \tilde{\epsilon} - \tilde{z}_4)$ crosses the contour from right to left. To compensate we have to *subtract* its residue and obtain

$$\begin{aligned} F_{\mathrm{OSB}}^{(\frac{1}{4}<\tilde{\epsilon}<\frac{1}{2})}\left(\vec{1}\right) =& \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)} \left[\frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(\tilde{z}_2)=0} d\tilde{z}_2 \int_{\mathrm{Re}(\tilde{z}_4)=0} d\tilde{z}_4 \, f_{\mathrm{OSB}}(\tilde{\epsilon}, \tilde{z}_2, \tilde{z}_4), \right. \\ & \left. - \frac{1}{2\pi i} \int_{\mathrm{Re}(\tilde{z}_2)=0} d\tilde{z}_2 \, \mathrm{Res}_{\tilde{z}_4=1/4-\tilde{\epsilon}} \left(f_{\mathrm{OSB}}(\tilde{\epsilon}, \tilde{z}_2, \tilde{z}_4) \right) \right]\,, \end{aligned} \tag{4.21}$$

$$
\begin{aligned}
\mathrm{Res}_{\tilde{z}_4=1/4-\tilde{\epsilon}}\left(f_{\mathrm{OSB}}\right)= & -\frac{(-M^2)^{-\frac{1}{2}+\tilde{z}_2}(-t)^{-\tilde{\epsilon}}}{(-s)^{-\frac{1}{2}+\tilde{z}_2-\tilde{\epsilon}}}\Gamma\left(\frac{1}{2}+\tilde{z}_2\right)\Gamma\left(\frac{1}{2}-\tilde{\epsilon}+\tilde{z}_2\right) \\
& \times\Gamma\left(1-\tilde{\epsilon}\right)\Gamma^2\left(\frac{1}{2}-\tilde{z}_2\right)\Gamma\left(\tilde{\epsilon}\right)\,. \qquad (4.22)
\end{aligned}
$$

Next, as $\tilde{\epsilon}$ crosses $1/2$ the 'right-most' pole of $\Gamma\left(\frac{1}{2}-\tilde{\epsilon}+\tilde{z}_2\right)$ in the one-dimensional MB integral crosses the contour from left to right. *Adding* its residue to the one-dimensional integral we get

$$
\begin{aligned}
F_{\mathrm{OSB}}^{(\frac{1}{2}<\tilde{\epsilon}<\frac{3}{4})}\left(\vec{1}\right)= & \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)}\left[\frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(\tilde{z}_2)=0}d\tilde{z}_2\int_{\mathrm{Re}(\tilde{z}_4)=0}d\tilde{z}_4\, f_{\mathrm{OSB}}(\tilde{\epsilon},\tilde{z}_2,\tilde{z}_4),\right. \\
& -\left[\frac{1}{2\pi i}\int_{\mathrm{Re}(\tilde{z}_2)=0}d\tilde{z}_2\,\mathrm{Res}_{\tilde{z}_4=1/4-\tilde{\epsilon}}\left(f_{\mathrm{OSB}}(\tilde{\epsilon},\tilde{z}_2,\tilde{z}_4)\right)\right. \\
& \left.\left.+\mathrm{Res}_{\tilde{z}_2=-1/2+\tilde{\epsilon}}\left(\mathrm{Res}_{\tilde{z}_4=1/4-\tilde{\epsilon}}\left(f_{\mathrm{OSB}}(\tilde{\epsilon},\tilde{z}_2,\tilde{z}_4)\right)\right)\right]\right], \qquad (4.23)
\end{aligned}
$$

$$
\mathrm{Res}_{\substack{\tilde{z}_4=1/4-\tilde{\epsilon}\\ \tilde{z}_2=-1/2+\tilde{\epsilon}}}\left(f_{\mathrm{OSB}}\right)=-(-M^2)^{-1+\tilde{\epsilon}}(-t)^{-\tilde{\epsilon}}(-s)\Gamma^3(1-\tilde{\epsilon})\Gamma^2(\tilde{\epsilon})\,. \qquad (4.24)
$$

Finally, the 'left-most' pole of $\Gamma\left(\frac{3}{4}-\tilde{\epsilon}-\tilde{z}_{24}\right)$ is crossed from right to left as $\tilde{\epsilon}$ crosses $3/4$ and we obtain

$$
\begin{aligned}
F_{\mathrm{OSB}}^{(\frac{3}{4}<\tilde{\epsilon}<\frac{3}{2})}\left(\vec{1}\right)= & \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)}\left[\frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(\tilde{z}_2)=0}d\tilde{z}_2\int_{\mathrm{Re}(\tilde{z}_4)=0}d\tilde{z}_4\, f_{\mathrm{OSB}}(\tilde{\epsilon},\tilde{z}_2,\tilde{z}_4),\right. \\
& -\frac{1}{2\pi i}\int_{\mathrm{Re}(\tilde{z}_2)=0}d\tilde{z}_2\,\left[\mathrm{Res}_{\tilde{z}_4=3/4-\tilde{\epsilon}-\tilde{z}_2}\left(f_{\mathrm{OSB}}\right)+\mathrm{Res}_{\tilde{z}_4=1/4-\tilde{\epsilon}}\left(f_{\mathrm{OSB}}\right)\right] \\
& \left.-\mathrm{Res}_{\tilde{z}_2=-1/2+\tilde{\epsilon}}\left(\mathrm{Res}_{\tilde{z}_4=1/4-\tilde{\epsilon}}\left(f_{\mathrm{OSB}}(\tilde{\epsilon},\tilde{z}_2,\tilde{z}_4)\right)\right)\right], \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Res}_{\tilde{z}_4=3/4-\tilde{\epsilon}-\tilde{z}_2}\left(f_{\mathrm{OSB}}\right)= & -(-M^2)^{-\frac{1}{2}+\tilde{z}_2}(-t)^{\frac{1}{2}-\tilde{\epsilon}-\tilde{z}_2}(-s)^{\tilde{\epsilon}}\Gamma\left(1-\tilde{\epsilon}\right)\Gamma\left(\frac{3}{2}-\tilde{\epsilon}-\tilde{z}_2\right) \\
& \times\Gamma\left(-\frac{1}{2}+\tilde{z}_2\right)\Gamma\left(\frac{1}{2}-\tilde{z}_2\right)\Gamma\left(-\frac{1}{2}+\tilde{\epsilon}+\tilde{z}_2\right)\,. \qquad (4.26)
\end{aligned}
$$

Now, the contours are fixed such that there is no pinching for $\tilde{\epsilon}\to 1$. Thus, we can switch back to the original variables and expand the integrands in ϵ after which they have to be solved up to the desired order. The two-dimensional MB integral only contributes at $\mathcal{O}(\epsilon)$ and we get up to the finite term

$$
\begin{aligned}
F_{\mathrm{OSB}}\left(\vec{1}\right)= & \frac{(-s)^{-2-\epsilon}}{\Gamma(-2\epsilon)}\left[(-M^2)^{\epsilon}(-t)^{-1-\epsilon}(-s)\Gamma^3(-\epsilon)\Gamma^2(1+\epsilon)\right] \\
& -\frac{1}{2\pi i}\int_{\mathrm{Re}(z_2)=-\frac{1}{2}}\frac{(-M^2)^{z_2}(-s)^{1-z_2}}{(-t)^{1+z_2}}\left[(-s)^{z_2}+(-t)^{z_2}\right]
\end{aligned}
$$

$$
\times \frac{\Gamma^2(-z_2)\Gamma(z_2)\Gamma(1+z_2)}{\epsilon} + \mathcal{O}(\epsilon^0)\Bigg]
$$

$$
= \frac{e^{-\epsilon\gamma_E}}{st}\left[\frac{2}{\epsilon^2} + \frac{2(\log(-M^2) - \log(-s) - \log(-t))}{\epsilon} - \frac{\pi^2}{2} + 2\log(-s)\log(-t)\right.
$$

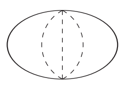
$$
\left. - \log^2(-M^2) - 2\mathrm{Li}_2\left(1 - \frac{M^2}{s}\right) - 2\mathrm{Li}_2\left(1 - \frac{M^2}{t}\right) + \mathcal{O}(\epsilon)\right]. \tag{4.27}
$$

There is a number of useful `Mathematica` packages that can be used when applying Mellin Barnes representations for Feynman integrals. `AMBRE` [14, 15] constructs optimized Mellin Barnes representations for certain classes of Feynman integrals and `MB` [16] and `MBresolve` [17] resolve singularities and perform expansions in ϵ or analytic regulators. The remaining Mellin Barnes integrals can be converted to multiple sums with `MBsums` [18], which can in turn be evaluated with `HarmonicSums` [19]. Alternatively, in Euclidean kinematics, the Mellin Barnes integrations can be performed numerically with the routines from `MB` [16].[1]

4.1 Exercises

1. Determine the leading term of the integral $F_{\text{Bhabha}}(1,1,0,1)$ defined below in Eq. (6.32) in an expansion for $t \ll m^2$ using Mellin Barnes representations.

2. Solve the 4-loop vacuum integral



$$
= \int \frac{[dl]_4}{[-l_1^2+1][-(l_1-l_2)^2][-(l_2-l_3)^2][-(l_3-l_4)^2][-l_4^2+1]} \tag{4.28}
$$

with mass set to one to all orders in ϵ.
Hint: Apply a Mellin Barnes representation at the propagator level that lets you evaluate all loop integrals by repeatedly applying the expression for the massless one-loop propagator integral.

Files: `Examples/MB_Box.m`, `Examples/MB_OffShellBox.m`,
`Exercises/MB_MassiveTriangleExpansion.m`

[1] See also [20] for ideas on numerical MB integration in Minkowski kinematics. At the time of writing the package `MBnumerics` [21] has been announced but is not yet available.

4.2 Solutions

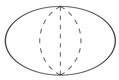
We apply

$$\frac{1}{[-l_4^2+1]} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(1+z)\Gamma(-z)}{[-l_4^2]^{1+z}} \tag{4.29}$$

and use the one-loop integral

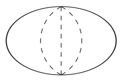
$$\int \frac{[dl]}{[-l^2]^\lambda[-(l+q)^2]} = \frac{\Gamma(1-\epsilon)\Gamma(\lambda-1+\epsilon)\Gamma(2-\lambda-\epsilon)}{\Gamma(\lambda)\Gamma(3-\lambda-2\epsilon)}[-q^2]^{1-\lambda-\epsilon} \tag{4.30}$$

repeatedly to obtain the simple MB representation



$$= \frac{\Gamma^3(1-\epsilon)}{\Gamma(2-\epsilon)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(-3+4\epsilon+z)\Gamma(-2+3\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z) \tag{4.31}$$

which can be solved by the first Barnes lemma with the result



$$= \frac{\Gamma^3(1-\epsilon)\Gamma(-1+2\epsilon)\Gamma(-2+3\epsilon)^2\Gamma(-3+4\epsilon)}{\Gamma(2-\epsilon)\Gamma(-4+6\epsilon)} \tag{4.32}$$

5 Reduction to master integrals with integration-by-parts identities

With increasing number of loops and legs we face a large number of Feynman diagrams each of which yields a large number of individual integrals of the type (1.1). These integrals are not independent but related through integration-by-parts identities [22]. In dimensional regularization surface terms vanish and we obtain relations of the type

$$\int [dl]_L \frac{\partial}{\partial l_i^\mu} \left(\frac{k_j^\mu}{[P_1^2 - m_1^2 + i0]^{n_1} \dots [P_N^2 - m_N^2 + i0]^{n_N}} \right) = 0 \,, \tag{5.1}$$

where k_j is one of the loop or external momenta. The expression on the left-hand side can be rewritten in terms of integrals of the same *family* with different values of the indices n_k. These identities allow us to reduce the large number of integrals to a (typically) much smaller set of basis or master integrals.

5.1 Integration-by-parts relations and reduction by hand

For illustration of the basic idea we consider the simple example of the massive 1-loop propagator integral

$$F_m(a_1, a_2) = \int [dl] \frac{1}{[l^2 - m^2]^{a_1} [(l-q)^2]^{a_2}} \tag{5.2}$$

The first IBP identity takes the form

$$\begin{aligned} 0 = &\int [dl] \frac{\partial}{\partial l^\mu} \frac{l^\mu}{[l^2 - m^2]^{a_1} [(l-q)^2]^{a_2}} \\ =& dF_m(a_1, a_2) - a_1 \int \frac{[dl]\, 2l^2}{[l^2 - m^2]^{a_1+1} [(l-q)^2]^{a_2}} - a_2 \int \frac{[dl]\, (2l^2 - 2l \cdot q)}{[l^2 - m^2]^{a_1} [(l-q)^2]^{a_2+1}} \\ =& [d - 2a_1 - a_2] F_m(a_1, a_2) - 2a_1 m^2 F_m(a_1+1, a_2) - a_2 F_m(a_1 - 1, a_2 + 1) \\ &+ a_2 (q^2 - m^2) F_m(a_1, a_2 + 1) \,. \end{aligned} \tag{5.3}$$

Eq. (5.3) can be written more compactly in operator form

$$d - 2a_1 - a_2 - 2a_1 m^2 \mathbf{1}^+ - a_2 \mathbf{2}^+ [\mathbf{1}^- - q^2 + m^2] = 0 \,, \tag{5.4}$$

where $\mathbf{n}^\pm F(\{a_j\}) = F(a_1, \dots, a_n \pm 1, \dots)$. The relation (5.4) by itself does not allow us to reduce the total power of propagators $a_1 + a_2$ and is therefore not sufficient to perform the reduction. The second IBP identity based on $\partial/(\partial l^\mu) q^\mu/(\dots)$ takes the form

$$a_2 - a_1 - a_1 \mathbf{1}^+ [q^2 + m^2 - \mathbf{2}^-] - a_2 \mathbf{2}^+ [\mathbf{1}^- - q^2 + m^2] = 0 \,. \tag{5.5}$$

Again, this is not very useful by itself but we can find a certain linear combination $(q^2 + m^2) \times (5.4) - 2m^2 \times (5.5)$ such that the $\mathbf{1}^+$ term cancels which gives

$$(q^2 - m^2)^2 a_2 \mathbf{2}^+ = - (d - 2a_1 - a_2) q^2 - (d - 3a_2) m^2$$

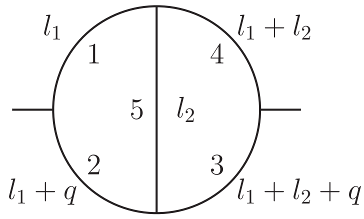


Figure 10: Massless two-loop propagator diagram.

$$+ (q^2 - m^2)a_2\mathbf{1}^-\mathbf{2}^+ + 2m^2a_1\mathbf{1}^+\mathbf{2}^- \,. \tag{5.6}$$

We observe that the application of (5.6) to the integral $F_m(a_1, a_2)$ provides an expression for $F_m(a_1, a_2 + 1)$ in terms of integrals with the total power of propagators reduced by one. E.g. setting $a_1 = a_2 = 1$ we find

$$\begin{aligned} F_m(1,2) =& \frac{1}{(q^2 - m^2)^2}\Bigg[- (d-3)(q^2 + m^2)F_m(1,1) \\ &+ (q^2 - m^2)F_m(0,2) + 2m^2F_m(2,0)\Bigg] \,, \end{aligned} \tag{5.7}$$

which can be simplified further by noting that (5.2) with $a_1 \leq 0$ is scaleless and thus vanishes in dimensional regularization. We can now reduce any integral (5.2) with positive indices to a sum over integrals with $a_2 = 0, 1$ through repeated application of (5.6). Then we can apply (5.4) to shuffle powers from the first to the second propagator, use (5.6) to reduce to total propagator power and repeat these steps until an arbitrary integral $F_m(a_1, a_2)$ has been expressed in terms of the two master integrals $F_m(1,1)$ and $F_m(1,0)$.

Now, let us revisit the two-loop example of the massless two-loop propagator integral shown in Figure 10 which illustrates the power of IBP reduction to simplify calculations:

$$F_{\oplus}(\{a_i\}) = \int \frac{[dl]_2}{[l_1^2]^{a_1}[(l_1 + q)^2]^{a_2}[(l_1 + l_2)^2]^{a_3}[(l_1 + l_2 + q)^2]^{a_4}[l_2]^{a_5}} \tag{5.8}$$

The IBP identity following from the derivative $\partial/(\partial l_2^\mu) l_2^\mu$ is given by

$$1 = \frac{1}{d - a_3 - a_4 - 2a_5}\left[a_3\mathbf{3}^+(\mathbf{5}^- - \mathbf{1}^-) - a_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{2}^-)\right] \,. \tag{5.9}$$

By applying (5.9) we reduce the sum $a_1 + a_2 + a_5$ of propagator powers in the left triangle by one. This can be repeated up to the point where the first of these indices goes to zero. For the simplest non-trivial example we find

$$\begin{aligned} &F_{\oplus}(1,1,1,1,1) \\ =&\frac{-1}{2\epsilon}\left[F_{\oplus}(1,1,2,1,0) - F_{\oplus}(0,1,2,1,1) + F_{\oplus}(1,1,1,2,0) - F_{\oplus}(1,0,1,2,1)\right] \end{aligned}$$

$$=\frac{1}{\epsilon}\left[F_{\oplus}(2,1,1,0,1)-F_{\oplus}(2,1,1,1,0)\right], \tag{5.10}$$

where we have used the symmetries of the integral family

$$F_{\oplus}(a_1,a_2,a_3,a_4,a_5)=F_{\oplus}(a_2,a_1,a_4,a_3,a_5), \tag{5.11}$$

$$F_{\oplus}(a_1,a_2,a_3,a_4,a_5)=F_{\oplus}(a_3,a_4,a_1,a_2,a_5), \tag{5.12}$$

to simplify the result. The integrals with $a_5 = 0$ factorize into the product of two massless one-loop two-point integrals

$$\int\frac{[dl]}{[l^2]^a[(l+q)^2]^b}=(-1)^{-a-b}(-q^2)^{d/2-a-b}G(a,b), \tag{5.13}$$

where

$$G(a,b)=\frac{\Gamma(a+b-d/2)\Gamma(d/2-a)\Gamma(d/2-b)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}. \tag{5.14}$$

The other integral with $a_4 = 0$ does not directly factorize but is recursively one-loop. We can first perform the l_2 integration by means of (5.13) which only modifies the index of the propagator l_1^2 and changes the prefactor. Then the remaining l_1 integral is again of the form (5.13) and we obtain a result for (5.10) in terms of Gamma functions without ever performing a 'real' two-loop calculation. For our example we find

$$\begin{aligned}(5.10)=&\frac{(-q^2)^{-1-2\epsilon}}{\epsilon}G(1,1)[G(2,1)-G(2,1+\epsilon)]\\ =&-\frac{e^{-2\gamma_E\epsilon}}{(-q^2)^{1+2\epsilon}}\left[6\zeta(3)+\left(\frac{\pi^4}{10}+12\zeta(3)\right)\epsilon\right.\\ &\left.+\left(\frac{\pi^4}{5}+(24-\pi^2)\zeta(3)+42\zeta(5)\right)\epsilon^2+\mathcal{O}(\epsilon^3)\right].\end{aligned} \tag{5.15}$$

5.2 Systematic approach to IBP reduction

An IBP reduction is performed in almost all state-of-the art calculations of Feynman diagrams and is often one of the bottlenecks of calculations with run-times of up to half a year on machines with up to 1 TB of Ram. Given the complexity of the problem, it calls for efficient automation which was achieved with the Laporta algorithm [23] which is briefly sketched here.

We consider an integral *family* or *topology*

$$F(\{a_i\})=\int[dl]_L\,f_{\{a_i\}}(\{l_j\})=\int\frac{[dl]_L}{\prod_{j=1}^N D_j^{a_j}}, \tag{5.16}$$

where D_j depend linearly or quadratically on the loop momenta and are linearly independent. We need a total of $N = LN_p + L(L+1)/2$ independent denominators, where

N_p is the number of external momenta to be able to express all scalar products uniquely in terms of the D_j. Then we can express the set of $L(L+N_p)$ IBP relations

$$\int [dl]_L \frac{\partial}{\partial l_i^\mu} \left(\frac{k_j^\mu}{\prod_{j=1}^N D_j^{a_j}} \right) = 0\,, \tag{5.17}$$

where k_j can be one of the loop or external momenta, for each of the sets $\{a_i\}$ in the form

$$\sum_i c_i F(a_1 + b_{i,1}, \ldots, a_N + b_{i,N}) = 0\,. \tag{5.18}$$

In addition we have boundary conditions of the form

$$\forall \{a_i\},\; a_j \leq 0 \text{ for all } j \in J :\; F(\{a_i\}) = 0\,, \tag{5.19}$$

for certain sets J when integrals become scaleless and symmetry relations

$$F(a_1, \ldots, a_N) = (-1)^{\sum d_i a_i} F(a_{\pi(1)}, \ldots, a_{\pi(N)})\,, \tag{5.20}$$

for some permutation π of the indices with the constants d_i being either zero or one. In total the conditions (5.18), (5.19) and (5.20) provide a homogeneous and linear system of equations for the integrals which is under-determined because some integrals are irreducible and need to be provided from an explicit calculation. The system is infinite-dimensional but the number of irreducible integrals is always finite [24]. However, we have the freedom to choose a basis of irreducible or master integrals in terms of which the other integrals can then be expressed. Since the master integrals have to be computed explicitly we should choose this basis based on simplicity of the integrals. To make this suitable for automation we define an ordering between all integrals.

First we split the integrals into *sectors*

$$\sigma_\nu = \{(a_1, \ldots, a_N) :\; (a_i - 1/2) d_i > 0\} \tag{5.21}$$

of the directions $\nu = d_1, \ldots, d_N$ with $d_i = \pm 1$. Within a given sector each index is always either positive or non-positive. Sectors defined by a direction with smaller $\sum d_i$ contain less different denominators and are therefore considered simpler and we consider all integrals in sector ν_1 to be simpler than all integrals in sector ν_2, when ν_1 is simpler than ν_2. Within a given sector we consider an integral $F(\{a_i\})$ to be simpler than $F(\{b_i\})$ if the sum of its indices is smaller $\sum_i |a_i| < \sum_i |b_i|$, which provides a full ordering of the integrals.

The aim of the Laporta algorithm is to build a reduction table, meaning a list of identities that expresses integrals with the propagator powers being less than some cutoff in terms of the master integrals. This is done by processing IBP identities one by one. Given some $\{a_i\}$ and $\{l_i, k_j\}$ the corresponding IBP (5.18) is created and the identities that are already known are substituted in. Then the IBP is solved for the most complicated integral according to the ordering defined above, thus expressing a complicated

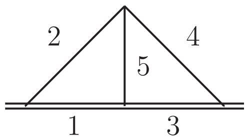


Figure 11: Massless two-loop propagator diagram.

integral in terms of simpler ones (not necessarily master integrals). Assuming the most complicated integral to have index k the identity takes the form

$$F(a_1 + b_{k,1}, \ldots, a_N + b_{k,N}) = -\sum_{i\neq k} \frac{c_i}{c_k} F(a_1 + b_{i,1}, \ldots, a_N + b_{i,N}) \,, \tag{5.22}$$

and is added to the system. At the same time (5.22) is substituted in the previously known identities.

What is left is to choose an efficient order in which the IBPs are processed. We must keep in mind that the list of identities can be huge and substituting in an identity for a simple integral that appears in many places at a late stage costs a lot of computing time. Thus, the choice of Laporta is to process the IBPs starting with the simplest set of indices. More specifically, we start in the simplest non-trivial sector without any extra powers, then subsequently add powers up to some cutoff and move on to the next complicated sector. The exact algorithm is laid out in detail in [23]. One of the first benchmarks for the Laporta algorithm was a non-planar 3-loop diagram for $g-2$ where about 100000 identities were processed and where computing time was reduced from several months to below an hour [23].

Today, there are many public codes based on the Laporta algorithm and its extensions which are well tested and able to tackle state-of-the art problems. In order of the appearance of the first public version those are `AIR` [25] (`Maple`), `FIRE` [26, 27] (`Mathematica` and `C++`), `Reduze` [28] (`C++`) and `Kira` [29] (`C++`). In addition there is a public implementation `LiteRed` [30] (`Mathematica`) of a different strategy for solving IBPs.

5.3 Exercises

1. Derive an IBP relation that allows you to compute the two-loop HQET propagator integral shown in Figure 11 with indices equal to one without performing any 'real' two-loop integration. The double lines denote linear propagators $1/(2v \cdot p)$ with $v^2 = 1$. Similar to the second example in Section 5.1 you can use that

$$\int \frac{[dl]}{[2v \cdot (l+q)]^a [l^2]^b} = [-2v \cdot q]^{d-a-2b} G_{\mathrm{HQET}}(a,b) \tag{5.23}$$

which follows from dimensional analysis.

5.4 Solutions

1. With the momentum assignment

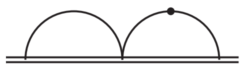
$$F(\{a_i\}) = \int \frac{[dl]_2}{[2v\cdot l_1 + 2\omega]^{a_1}[l_1^2]^{a_2}[2v\cdot(l_1+l_2)+2\omega]^{a_3}[(l_1+l_2)^2]^{a_4}[l_2^2]^{a_5}}, \quad (5.24)$$

where $\omega = v\cdot q$, we obtain the desired IBP identity from $\partial/(\partial l_2^\mu) l_2^\mu$

$$1 = \frac{1}{d - a_3 - a_4 - 2a_5}\left[-a_3\mathbf{3}^+\mathbf{1}^- + a_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{2}^-)\right]. \quad (5.25)$$

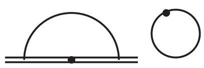
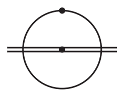
We obtain

$$F(1,1,1,1,1) = \frac{-1}{2\epsilon}\left[-F(0,1,2,1,1) + F(1,1,1,2,0) - F(1,0,1,2,1)\right] \quad (5.26)$$

The integral $F(1,1,1,2,0) =$  is already factorized. For the other ones we apply the symmetry under $(a_1,a_2) \leftrightarrow (a_3,a_4)$ and again make use of (5.25)

$$\begin{aligned} F(0,1,2,1,1) =& F(2,1,0,1,1) = \frac{1}{1-2\epsilon}\left[F(2,1,0,2,0) - F(2,0,0,2,1)\right] \\ F(1,0,1,2,1) =& F(1,2,1,0,1) = \frac{F(0,2,2,0,1)}{1-2\epsilon} = \frac{F(2,0,0,2,1)}{1-2\epsilon} \end{aligned}$$

The two remaining integrals with 3 propagators have the form

$F(2,1,0,2,0) =$  $= 0, \qquad F(2,0,0,2,1) =$  . (5.27)

The latter integral is recursively one-loop by means of (5.23).

Files: `Examples/IBP_Ex1.m`, `Examples/IBP_Toy_Laporta.m`, Examples with `FIRE`: `Exercises/IBP_TwoLoopProp.m`, `Exercises/IBP_TwoLoopPropHQET.m`

6 Master integrals from differential equations

The idea of using differential equations to evaluate master integrals dates back to 1991 [31] and has been very successful in recent years due to the ideas presented in [32, 33]. To apply this method to a given integral family it is necessary to set up the IBP reduction discussed in Section 5 and to identify the master integrals in this family. Writing them as entries of a N-vector $\vec{f}$ we can take the partial derivatives of this vector with respect to the M kinematic invariants and masses in the system, denoted as $\{X_i\}$. This gives sums over integrals in the same family with shifted indices which can be reduced to the master integrals by IBP reduction. Thus, we find coupled systems of linear differential equations of the form

$$\partial_i \vec{f}(\epsilon, \{X_j\}) = A_i(\epsilon, \{X_j\})\, \vec{f}(\epsilon, \{X_j\})\,, \qquad i = 1, \ldots, M\,, \tag{6.1}$$

where $\partial_i = \partial/(\partial X_i)$ and A_i is a $N \times N$ matrix whose entries are rational functions of ϵ and the invariants X_j due to the structure of the IBPs. In the more compact notation of differential forms this takes the form

$$\mathrm{d}\vec{f}(\epsilon, \{X_j\}) = A(\epsilon, \{X_j\})\, \vec{f}(\epsilon, \{X_j\})\,, \tag{6.2}$$

where $\mathrm{d}\vec{f} = \sum_i (\partial \vec{f})/(\partial X_i)\, \mathrm{d}X_i$ and $A(\epsilon, \{X_j\}) = \sum_i A_i(\epsilon, \{X_j\})\, \mathrm{d}X_i$. In general the differential equations (6.1) are very difficult to solve. Therefore, it was proposed in [32] to perform a basis change $\vec{f} = T\vec{g}$ to a *canonical form* where

$$\mathrm{d}\vec{g}(\epsilon, \{X_j\}) = \epsilon\, \mathrm{d}\tilde{A}(\{X_j\})\, \vec{g}(\epsilon, \{X_j\})\,, \tag{6.3}$$

where

$$\tilde{A}(\{X_j\}) = \sum_{k=1}^{K} \tilde{A}_k \log(L_k(\{X_j\}))\,. \tag{6.4}$$

The $L_k(\{X_j\})$ are called *letters* of the *alphabet*

$$\mathcal{A} = \{L_1(\{X_j\}), \ldots, L_K(\{X_j\})\}\,, \tag{6.5}$$

and the alphabet determines the function space of the solution.

The formal solution to (6.3) then takes the form of Chen iterated integrals [34]

$$\vec{g}(\epsilon, \{X_j\}) = \mathbb{P} \exp\left[\epsilon \int_\gamma \mathrm{d}\tilde{A}\right] \vec{g}(\epsilon, \{X_{j0}\})\,, \tag{6.6}$$

where $\vec{g}(\{X_{j0}\})$ is some boundary value, γ is a path from $\{X_{j0}\}$ to $\{X_j\}$ and $\mathbb{P}$ denotes path ordering along γ. In praxis we are mainly interested in the expansion of $\vec{g}$ up to some power in ϵ. We use the invariance of the differential equation (6.3) under the rescaling $\vec{g} \to \epsilon^n \vec{g}$ to make the basis integrals finite and write without loss of generality

$$\vec{g}(\epsilon, \{X_j\}) = \sum_{n=0}^{\infty} \epsilon^n\, \vec{g}^{(n)}(\{X_j\})\,. \tag{6.7}$$

Then the solution to (6.3) takes the form

$$\vec{g}^{(0)}(\{X_j\}) = \vec{g}^{(0)}(\{X_{j0}\}), \tag{6.8}$$

$$\vec{g}^{(n)}(\{X_j\}) = \int_\gamma \mathrm{d}\tilde{A}(\{X_j\})\,\vec{g}^{(n-1)}(\{X_j\}) + \vec{g}^{(n)}(\{X_{j0}\}), \tag{6.9}$$

where γ again denotes a path from $\{X_{j0}\}$ to $\{X_j\}$. Choosing a specific integration path (typically piecewise linear) the solution for a system with a rational alphabet can be written in terms of Goncharov polylogarithms [35] which are defined recursively as

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; z)\,, \tag{6.10}$$

with $G(z) = 1$ and with the special case

$$G(\vec{0}_n; z) = \frac{1}{n!} \log^n(z)\,, \tag{6.11}$$

when all indices a_i are zero. We call the number n of iterated integrations the *weight*. The integrals (6.10) are related to the multiple polylogarithms (MPL)

$$\mathrm{Li}_{m_1,\ldots,m_k}(z_1, \ldots, z_k) = \sum_{0<n_1<n_2<\cdots<n_k} \frac{z_1^{n_1} z_2^{n_2} \ldots z_k^{n_k}}{n_1^{m_1} n_2^{m_2} \ldots n_k^{m_k}} \tag{6.12}$$

by the identity

$$\mathrm{Li}_{m_1,\ldots,m_k}(z_1, \ldots, z_k) = (-1)^k G(\underbrace{0, \ldots, 0}_{m_k-1}, \frac{1}{z_k}, \ldots, \underbrace{0, \ldots, 0}_{m_1-1}, \frac{1}{z_1 \ldots z_k}; 1). \tag{6.13}$$

For their properties we refer to the lecture notes on MPLs [36]. There also exists a library for the evaluation of multiple polylogarithms to arbitrary precision [37].

Let us illustrate the solution in terms of MPLs for a simple case with just one master integral g, one variable x, the boundary condition $g(0) = g_0$ and the 'matrix' $\tilde{A} = a_1 \log(x - x_1) + a_2 \log(x - x_2)$. Then we obtain the solution

$$g^{(0)}(x) = g_0^{(0)}\,, \tag{6.14}$$

from (6.8). At order ϵ we obtain using (6.9)

$$\begin{aligned} g^{(1)}(x) =& \int_0^x dt \left(\frac{a_1}{t - x_1} + \frac{a_2}{t - x_2} \right) g^{(0)}(t) + g_0^{(1)} \\ =& a_1 g_0^{(0)} G(x_1; x) + a_2 g_0^{(0)} G(x_2; x) + g_0^{(1)}\,. \end{aligned} \tag{6.15}$$

At higher orders in ϵ we simply obtain Goncharov polylogarithms of higher weight

$$g^{(2)}(x) = \int_0^x dt \left(\frac{a_1}{t - x_1} + \frac{a_2}{t - x_2} \right) \left[a_1 g_0^{(0)} G(x_1; t) + a_2 g_0^{(0)} G(x_2; t) + g_0^{(1)} \right] + g_0^{(2)}$$

$$
\begin{aligned}
=&g_0^{(0)}\left[a_1^2 G(x_1,x_1;x)+a_1a_2G(x_1,x_2;x)+a_1a_2G(x_2,x_1;x)+a_2^2G(x_2,x_2;x)\right]\\
&+g_0^{(1)}\left[a_1G(x_1;x)+a_2G(x_2;x)\right]+g_0^{(2)}\,, \qquad (6.16)
\end{aligned}
$$

such that solving the differential equations (6.3) in canonical form is essentially reduced to a combinatorical problem and the determination of the boundary values.

Clearly the crucial step in the differential equations method is to find a basis transformation that bring the differential equation into the *canonical form*. Early discussions of this subject can be found in [38, 33]. More recently, public codes that can find these transformations have appeared. To date there are `epsilon` [39] and `Fuchsia` [40], which are based on an algorithm proposed in [41], and `CANONICA` [42] which is based on [43] and also works with multiple scales. We will not discuss these ideas here however.

Let us revisit the massless box diagram (4.8) we solved using a Mellin Barnes representation as a simple example. We make our basis dimensionless by rescaling with appropriate powers of $(-s)$ such that our naive basis integrals

$$
\vec{f}(\epsilon,x)=\begin{pmatrix}(-s)^{\epsilon}F_{\Box}(0,1,0,1)\\ (-s)^{\epsilon}F_{\Box}(1,0,1,0)\\ (-s)^{2+\epsilon}F_{\Box}(1,1,1,1)\end{pmatrix} \qquad (6.17)
$$

only depend on one dimensionless ratio $x=t/s$. In this basis the differential equation takes the form

$$
\partial_x\vec{f}(\epsilon,x)=\begin{pmatrix}-\frac{\epsilon}{x} & 0 & 0\\ 0 & 0 & 0\\ \frac{2(2\epsilon-1)}{x}-\frac{2(2\epsilon-1)}{x^2}-\frac{2(2\epsilon-1)}{x+1} & \frac{2(2\epsilon-1)}{x}-\frac{2(2\epsilon-1)}{x+1} & \frac{-\epsilon-1}{x}+\frac{\epsilon}{x+1}\end{pmatrix}\vec{f}(\epsilon,x)\,, \qquad (6.18)
$$

which is clearly not in the form (6.3). Here, it is fairly straightforward to determine the transformation $\vec{f}=T\vec{g}$ to a canonical basis

$$
\vec{g}(\epsilon,x)=\epsilon e^{\epsilon\gamma_E}(-s)^{\epsilon}\begin{pmatrix}tF_{\Box}(0,1,0,2)\\ sF_{\Box}(1,0,2,0)\\ \epsilon stF_{\Box}(1,1,1,1)\end{pmatrix}=\epsilon e^{\epsilon\gamma_E}(-s)^{\epsilon}\begin{pmatrix}-\frac{F_{\Box}(0,1,0,1)}{1-2\epsilon}\\ -\frac{F_{\Box}(1,0,1,0)}{1-2\epsilon}\\ \epsilon stF_{\Box}(1,1,1,1)\end{pmatrix} \qquad (6.19)
$$

where the matrix A transforms as

$$
A(\epsilon,x)\to T^{-1}\left[A(\epsilon,x)T-\partial_xT\right]\,. \qquad (6.20)
$$

The constant prefactor $\epsilon e^{\epsilon\gamma_E}$ in (6.19) does not affect the matrix A and is chosen like this to make the basis finite and to remove any powers of γ_E from the expansion of $\vec{g}$ in ϵ. In the new basis, the differential equation is in *canonical form*

$$
\partial_x\vec{g}(\epsilon,x)=\epsilon\left[\frac{a}{x}+\frac{b}{1+x}\right]\vec{g}(\epsilon,x)\,, \qquad (6.21)
$$

with

$$
a=\begin{pmatrix}-1&0&0\\0&0&0\\-2&0&-1\end{pmatrix},\qquad b=\begin{pmatrix}0&0&0\\0&0&0\\2&2&1\end{pmatrix}. \qquad (6.22)
$$

We find that the solution of (6.21) is trivial for g_1 and g_2

$$g_1(x) = x^\epsilon g_1(0), \qquad g_2(x) = g_2(0). \tag{6.23}$$

The reason is that g_1 and g_2 are single-scale integrals and the scaling behaviour can be determined solely from power counting. The differential equation thus provides no additional information and these integrals have to be solved using other means.[2] With Feynman parameters we obtain

$$g_1 = x^{-\epsilon} g_2\,, \qquad g_2 = -e^{\epsilon\gamma_E} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{6.24}$$

This commonly happens for some of the simplest master integrals in a given family which are typically already known in the literature. Finally, we need to determine a boundary value for g_3 at some convenient choice of $x = x_0$. This can in principle be done using some other method and is a significantly simpler calculation than the calculation of the full master integral due to the reduced number of scales. Here, and in many cases the boundary value is however supplied by the differential equation itself together with some knowledge about the analytic structure of the integral. Let us consider (6.21) in the limit $x \to -1$, i.e. $t \to -s$ which corresponds to $u = -s - t \to 0$. Keeping only the leading term in $1 + x$ we obtain the solution (noting that $b^2 = b$)

$$\lim_{x\to -1} \vec{g}(\epsilon, x) =(1+x)^{\epsilon b}\vec{g}(\epsilon,-1) = [1 + b(1+x)^\epsilon]\,\vec{g}(\epsilon,-1)\,. \tag{6.25}$$

Expanding in ϵ we find that the solution develops an imaginary part starting at $x \le -1$ from some order in ϵ unless $b\vec{g}(\epsilon,-1) = \vec{0}$ to all orders in ϵ. We know however that imaginary parts are related to on-shell cuts and no such u-channel cut exists for the box diagram. Thus we obtain the following boundary condition with no need for an explicit calculation

$$\begin{aligned} g_3(\epsilon,-1+i0) =& -2[g_1(\epsilon,-1+i0) + g_2(\epsilon,-1+i0)] & (6.26)\\ =&4 - 2i\pi\epsilon - \frac{4\pi^2}{3}\epsilon^2 + \left(\frac{i\pi^3}{2} - \frac{28\zeta(3)}{3}\right)\epsilon^3 + \mathcal{O}(\epsilon^4)\,. & (6.27)\end{aligned}$$

Now we can solve the differential equation (6.21) for g_3

$$\begin{aligned} g_3^{(0)}(x) =& g_3^{(0)}(-1) = 4\,, & (6.28)\\ g_3^{(1)}(x) =& \int_0^x dt \left(\frac{(-2,0,-1)}{t} + \frac{(2,2,1)}{1+t}\right)\begin{pmatrix}-1\\-1\\4\end{pmatrix} + g_3^{(1)}(0) & \\ =& -2G(0;x) + g_3^{(1)}(0) = -2\log(x)\,, & (6.29)\end{aligned}$$

---

[2] It is however possible to introduce an auxiliary scale and to use the ability of the differential equation to provide information on the boundary condition which corresponds to the desired integral [44].

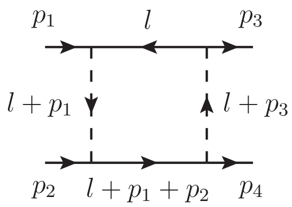


Figure 12: One-loop Bhabha scattering.

$$
\begin{aligned}
g_3^{(2)}(x) =& \int_0^x dt \left(\frac{(-2,0,-1)}{t} + \frac{(2,2,1)}{1+t} \right) \begin{pmatrix} G(0;x) \\ 0 \\ -2G(0;x) \end{pmatrix} + g_3^{(2)}(0) \\
=& g_3^{(2)}(0) = -\frac{4\pi^2}{3} \,, \qquad (6.30) \\
g_3^{(3)}(x) =& \int_0^x dt \left(\frac{(-2,0,-1)}{t} + \frac{(2,2,1)}{1+t} \right) \begin{pmatrix} \pi^2/12 - G(0,0;x) \\ \pi^2/12 \\ -4\pi^2/3 \end{pmatrix} + g_3^{(3)}(0) \\
=& \frac{7\pi^2}{6} G(0;x) + 2G(0,0,0;x) - \pi^2 G(-1;x) - 2G(-1,0,0;x) + g_3^{(3)}(0) \\
=& \frac{7\pi^2}{6} \ln(x) + \frac{1}{3} \ln^3(x) - \pi^2 \ln(1+x) \\
& - 2\left[\frac{1}{2} \ln^2(x) \ln(1+x) + \ln(x) \mathrm{Li}_2(-x) - \mathrm{Li}_3(-x) \right] - \frac{34}{3}\zeta(3) \,, \qquad (6.31)
\end{aligned}
$$

which reproduces the result (4.17) obtained in Section 4. We observe that, assigning weight one to π and weight n to $\zeta(n)$, all terms in $\vec{g}^{(n)}$ have weight n. This property follows direct from the structure of the differential equation in canonical form and its solution in terms of iterated integrals. It can be helpful in identifying possible elements of the canonical basis.

Now let us discuss an example with two ratios of scales, the one-loop Bhabha scattering $e^+e^- \to e^+e^-$ considered in [45]. The integral family of Figure 12 is given by

$$
F_{\text{Bhabha}}(a_1, \ldots, a_4) = \int \frac{[dl]}{[l^2 - m^2]^{a_1} [(l+p_1)^2]^{a_2} [(l+p_1+p_2)^2 - m^2]^{a_3} [(l+p_3)^2]^{a_4}} \,, \tag{6.32}
$$

where $p_i^2 = m^2$, $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$ and $u = (p_2-p_3)^2$ with $s+t+u = 4m^2$. Due to the threshold at $s = 4m^2$ expressions of the form $\sqrt{(4m^2-s)(-s)}$ appear. To obtain a rational alphabet we use the variables x, y defined as

$$
-\frac{s}{m^2} = \frac{(1-x)^2}{x}, \qquad -\frac{y}{m^2} = \frac{(1-y)^2}{y}, \tag{6.33}
$$

which covert these roots to rational expressions. The points $x = -1, 0, 1$ correspond to $s = 4m^2, \infty, 0$ with the same relation between y and t and $x = -y$ corresponds to $u = 0$.

The canonical basis takes the form [45][3]

$$\vec{f}(\epsilon,x,y) = (m^2)^\epsilon \epsilon e^{\epsilon\gamma_E} \begin{pmatrix} F_{\text{Bhabha}}(2,0,0,0) \\ -tF_{\text{Bhabha}}(0,2,0,1) \\ -\sqrt{(4m^2-s)(-s)}F_{\text{Bhabha}}(2,0,1,0) \\ 2\epsilon\sqrt{(4m^2-t)(-t)}F_{\text{Bhabha}}(1,1,0,1) \\ -2\epsilon\sqrt{(4m^2-s)(-s)}tF_{\text{Bhabha}}(1,1,1,1) \end{pmatrix}, \tag{6.34}$$

with the differential equation given by (6.3) with

$$\begin{aligned}\tilde{A} =& a_1\log(x) + a_2\log(1+x) + a_3\log(y) + a_4\log(1+y) + a_5\log(1-y) \\ &+ a_6\log(x+y) + a_7\log(1+xy)\,,\end{aligned} \tag{6.35}$$

where the matrices a_i can be read off from Eq. (2.14) of [45]. The two simple master integrals take the form

$$f_1(\epsilon,x,y) = \epsilon e^{\epsilon\gamma_E}\Gamma(\epsilon)\,, \tag{6.36}$$

$$f_2(\epsilon,x,y) = -\,\epsilon e^{\epsilon\gamma_E}\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{y}{(1-y)^2}\right)^\epsilon. \tag{6.37}$$

For the other elements we have the following boundary conditions

$$f_3(\epsilon,x=1,y) = 0\,, \tag{6.38}$$

$$f_4(\epsilon,x,y\to 1) = \epsilon e^{\epsilon\gamma_E}\frac{2^{1+2\epsilon}\pi\Gamma(1/2-\epsilon)\Gamma(1/2+\epsilon)}{\Gamma(-\epsilon)}(1-y)^{-2\epsilon} + \mathcal{O}\left((1-y)^{1/2}\right), \tag{6.39}$$

$$f_5(\epsilon,x=1,y) = 0\,, \tag{6.40}$$

which follow from $F_{\text{Bhabha}}(2,0,1,0)$, $F_{\text{Bhabha}}(1,1,1,1) \to$ const as $s\to 0$, i.e. $x\to 1$, and $F_{\text{Bhabha}}(1,1,0,1) \to \text{const} \times (-t)^{-1/2-\epsilon}$ as $t\to 0$, i.e. $y\to 1$, which was derived in the exercise 4.1. From (6.8) and the boundary conditions we obtain

$$\vec{f}^{(0)}(x,y) = (1,1,0,0,0)\,. \tag{6.41}$$

To obtain higher orders in ϵ we choose the integration path γ in (6.9) as piecewise linear from $(0,0)\to(x,0)$ and $(x,0)\to(x,y)$. We derive expressions valid in the Euclidean region $0<x<1$ and $0<y<1$ where the result is real. Expressions for the physical region can be obtained by analytic continuation. We find

$$\begin{aligned}\vec{f}^{(n)}(x,y) =& \vec{f}^{(n)}(0,0) + \int_0^x dt\left[\frac{a_1+a_6}{t} + \frac{a_2}{t+1}\right]\vec{f}^{(n-1)}(t,0) \\ &+ \int_0^y dt\left[\frac{a_3}{t} + \frac{a_4}{t+1} + \frac{a_5}{t-1} + \frac{a_6}{t+x} + \frac{a_7}{t+\frac{1}{x}}\right]\vec{f}^{(n-1)}(x,t)\,.\end{aligned} \tag{6.42}$$

[3] There is a type in Eq. (2.9) of [45] as is clear from the dimensionality.

As an example consider the contribution from the a_7 matrix to $\vec{f}^{(3)}$. With

$$a_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 \end{pmatrix} \tag{6.43}$$

and

$$\vec{f}^{(2)}(x,y) = \begin{pmatrix} \frac{\pi^2}{12} \\ G(0,0;y) - 2G(0,1;y) - 2G(1,0;y) + 4G(1,1;y) - \frac{\pi^2}{12} \\ 2G(-1,0;x) - G(0,0;x) + \frac{\pi^2}{6} \\ -2G(0,0;y) + 4G(0,1;y) - \frac{4\pi^2}{3} \\ 4G(0;x)G(0;y) - 8G(0;x)G(1;y) \end{pmatrix} \tag{6.44}$$

we find

$$\begin{aligned} \int_0^y \frac{dt}{t+\frac{1}{x}} a_7 \vec{f}^{(n-1)}(x,t) = \Bigg(& 0,0,0,0, \frac{10\pi^2}{3} G\left(-\frac{1}{x};y\right) + 8G(-1,0;x)G\left(-\frac{1}{x};y\right) \\ & - 4G(0,0;x)G\left(-\frac{1}{x};y\right) + 4G(0;x)G\left(-\frac{1}{x},0;y\right) - 8G(0;x)G\left(-\frac{1}{x},1;y\right) \\ & + 4G\left(-\frac{1}{x},0,0;y\right) - 8G\left(-\frac{1}{x},0,1,y\right)\Bigg). \end{aligned} \tag{6.45}$$

We see that the letter $1+xy$ first contributes at this order and only to f_5. The full result to order ϵ^3 is derived in the corresponding `Mathematica` file and is of uniform weight 3. We note that the specific form of the solution is not unique but depends on the chosen integration path. The Goncharov polylogarithms satisfy many relations and finding an 'optimal' representation of the solution is in general non-trivial. In a famous example, a 17-page long result [46, 47] for a two-loop six-point amplitude in $\mathcal{N}=4$ SYM in terms of weight four MPLs was expressed in two lines in terms of classical polylogarithms [48]. MPLs up to weight three can always be expressed in terms of classical polylogarithms with appropriate arguments. At weight four this is only possible if certain conditions are satisfied [48].

6.1 Exercises

1. Derive the canonical form of the differential equation for the example of one-loop Bhabha scattering considered above.

Files: `Examples/DE_Box.m`, `Examples/DE_Bhabha.m`

7 Expansion by regions

The expansion by regions [49, 50, 51] is a method to expand loop integrals in a small parameter at the *integrand level.* In Section 4 we already saw that we cannot obtain the leading behaviour of the massless one-loop box in $t \ll s$ by naively Taylor-expanding the integrand. To determine such expansions without performing the full calculation and then expanding the result one can apply MB techniques or the expansion by regions. The latter has the advantage of a much clearer physical interpretation and a close relation to effective field theories. We introduce the expansion by regions with the example [51] of a one-loop massive propagator integral

$$F(n_1, n_2) = \int \frac{[dl]}{[(l+q)^2]^{n_1}[l^2 - m^2]^{n_2}} \equiv \int [dl] I \,, \tag{7.1}$$

where we assume that $q^2 \gg m^2$, i.e. we want to determine F as an expansion in m^2/q^2. We can now introduce an intermediate scale Λ such that $q^2 \gg \Lambda^2 \gg m^2$ and divide the integration domain into a hard and soft domain

$$D_h = \{l \in \mathbb{R}^d \,:\, |l^2| \geq \Lambda^2\}, \qquad D_s = \{l \in \mathbb{R}^d \,:\, |l^2| < \Lambda^2\}. \tag{7.2}$$

In the hard domain we have $|l^2| \gg m^2$ and can therefore naively Taylor expand the integrand in m^2

$$I \overset{l \in D_h}{=} T^{(h)} I = \sum_{j=0}^{\infty} T_j^{(h)} I = \sum_{j=0}^{\infty} \frac{(n_2)_j}{j!} \frac{(m^2)^j}{[(l+q)^2]^{n_1}[l^2]^{n_2+j}} \,, \tag{7.3}$$

where $(n_2)_j = \Gamma(n_2 + j)/\Gamma(n_2)$. In the soft domain we expand as follows

$$I \overset{l \in D_s}{=} T^{(s)} I = \sum_{j=0}^{\infty} T_j^{(s)} I = \sum_{j_1, j_2} T_{j_1, j_2}^{(s)} I = \sum_{j_1, j_2 = 0}^{\infty} \frac{(n_1)_{j_1+j_2}}{j_1! j_2!} \frac{(-l^2)^{j_1}(-2l \cdot q)^{j_2}}{[q^2]^{n_1+j_1+j_2}[l^2 - m^2]^{n_2}} \,. \tag{7.4}$$

While $|l^2| \ll q^2$ holds in the soft domain, $|2l \cdot q|$ can be much larger than q^2 for kinematics where l is very energetic but nearly light-light. The expansion (7.4) is therefore only valid under the integral sign where we can perform a tensor reduction that shows that odd powers of $l \cdot q$ vanish and even powers yield terms proportional to $l^2 q^2 \sim m^2 q^2$. Thus, we can count $|l \cdot q| \sim \sqrt{m^2 q^2}$ and the index j counting the power of m^2 is given by $j = j_1 + j_2/2$. We say that the soft region has the scaling $l \sim m$ by which we mean that we expand according to the assumption that all components of l are of the size m. Similarly the hard region scales like $l \sim q$ meaning that all components of l are of the size of the overall momentum q. Now, we can split the original integral (7.1) in two parts

$$F(n_1, n_2) = \int_{D_h} [dl]\, T^{(h)} I + \int_{D_s} [dl]\, T^{(s)} I = \sum_j \int_{D_h} [dl]\, T_j^{(h)} I + \sum_j \int_{D_s} [dl]\, T_j^{(s)} I \,, \tag{7.5}$$

where we have used the absolute convergence of the expansions in their respective domains to exchange the order of the integration and summation. In dimensional regularization the integral of the expansions (7.3) and (7.4) over the whole domain is well defined and we find

$$F(n_1,n_2)=\sum_j\left[\int[dl]\,T_j^{(h)}I-\int_{D_s}[dl]\,T_j^{(h)}I\right]+\sum_j\left[\int[dl]\,T_j^{(s)}I-\int_{D_h}[dl]\,T_j^{(s)}I\right]. \tag{7.6}$$

In the hard and soft domain we may now again apply the respective expansions

$$\int_{D_s}[dl]\,T_j^{(h)}I=\sum_{k=0}^{\infty}\int_{D_s}[dl]\,T_k^{(s)}T_j^{(h)}I, \tag{7.7}$$

$$\int_{D_h}[dl]\,T_j^{(s)}I=\sum_{k=0}^{\infty}\int_{D_h}[dl]\,T_k^{(h)}T_j^{(s)}I. \tag{7.8}$$

The result of the double expansion does not depend on the order

$$T_{j,k}^{(h,s)}I\equiv T_{j_1,j_2}^{(s)}T_k^{(h)}I=T_k^{(h)}T_{j_1,j_2}^{(s)}I=\frac{(n_2)_k}{k!}\,\frac{(n_1)_{j_1+j_2}}{j_1!j_2!}\,\frac{(m^2)^k(-1)^{j_1}(-2l\cdot q)^{j_2}}{(q^2)^{n_1+j_1+j_2}(l^2)^{n_2+k-j_1}}, \tag{7.9}$$

and we obtain

$$\begin{aligned}F(n_1,n_2)=&\sum_j\int[dl]\,T_j^{(h)}I+\sum_j\int[dl]\,T_j^{(s)}I-\sum_{j,k}\int[dl]\,T_{j,k}^{(h,s)}I\\ \equiv&F^{(h)}(n_1,n_2)+F^{(s)}(n_1,n_2)-F^{(h,s)}(n_1,n_2)\,,\end{aligned} \tag{7.10}$$

where all contributions are now integrated over the whole integration domain and the dependence on the precise definition of the two domains has therefore canceled. Let us evaluate the expansion of the finite integral $F(1,2)$ by means of (7.10). The 'hard' and 'soft contributions' are

$$\begin{aligned}F_j^{(h)}(1,2)=&(1+j)(m^2)^j\int\frac{[dl]}{(l+q)^2[l^2]^{2+j}}\\ =&\frac{\left(-q^2-i0\right)^{-\epsilon}}{q^2}\left(\frac{m^2}{q^2}\right)^j\frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{(2\epsilon)_j}{j!}\,,\end{aligned} \tag{7.11}$$

$$\begin{aligned}F_j^{(s)}(1,2)=&\sum_{j_1=0}^{j}\sum_{j_2=0}^{2(j-j_1)}\frac{(j_1+j_2)!}{j_1!\,j_2!}\,\frac{(-1)^{j_1+j_2}}{(q^2)^{n_1+j_1+j_2}}\int\frac{[dl](l^2)^{j_1}\,(2l\cdot q)^{j_2}}{[l^2-m^2]^{n_2}}\\ =&\frac{\left(m^2\right)^{-\epsilon}}{q^2}\left(\frac{m^2}{q^2}\right)^j\,\Gamma(\epsilon)\,\frac{(\epsilon)_j}{(1-\epsilon)_j}\,.\end{aligned} \tag{7.12}$$

which can both be obtained using Feynman parameters. The 'overlap contribution' $F^{(h,s)}$ vanishes in dimensional regularization because the double expansion (7.9) produces only scaleless integrals. The contributions $F_j^{(h)}$ and $F_j^{(s)}$ are IR and UV divergent, respectively,

$$F_0^{(h)}(1,2)=\frac{1}{q^2}\left[-\frac{1}{\epsilon}+\ln\left(-q^2-i0\right)\right]+\mathcal{O}(\epsilon)\,,\quad F_j^{(h)}(1,2)=-\frac{2}{q^2}\left(\frac{m^2}{q^2}\right)^j\frac{1}{j}+\mathcal{O}(\epsilon)\,,$$

$$F_0^{(s)}(1,2) = \frac{1}{q^2}\left[\frac{1}{\epsilon} - \ln\left(m^2\right)\right] + \mathcal{O}(\epsilon)\,, \qquad F_j^{(s)}(1,2) = \frac{1}{q^2}\left(\frac{m^2}{q^2}\right)^j \frac{1}{j} + \mathcal{O}(\epsilon)\,, \tag{7.13}$$

but the $1/\epsilon$ pole cancels in the sum. Spurious singularities like that occur very often in the expansion by regions and their cancellation provides a good check on calculations. We note that the cancellation is not technically between UV and IR singularities because the scaleless overlap contribution is of the form $F^{(h,s)} \sim (1/\epsilon_{\text{UV}} - 1/\epsilon_{\text{IR}}) = 0$ and the IR and UV poles cancel in the combinations $F^{(h)} - F^{(h,s)}$ and $F^{(s)} - F^{(h,s)}$, respectively. We can sum up the result

$$F(1,2) = \sum_{j=0}^{\infty} F_j^{(h)}(1,2) + F_j^{(s)}(1,2) = \frac{1}{q^2}\left[\ln\left(\frac{-q^2 - i0}{m^2}\right) + \ln\left(1 - \frac{m^2}{q^2}\right)\right] + \mathcal{O}(\epsilon)\,, \tag{7.14}$$

which agrees with the direct evaluation of the integral.

Let us now generalize this to a situation where we have a decomposition of the integration domain into N domains D_{x_i} with associated commuting expansions $T^{(x_i)}$. Then, the expansion of the integral takes the form

$$F = \sum_{i_1=1}^{N} F^{(x_{i_1})} - \sum_{i_2 \neq i_1} F^{(x_{i_1}, x_{i_2})} + \cdots - (-1)^N F^{(x_1, \ldots, x_N)}\,, \tag{7.15}$$

where all integrations are performed over the complete integration domain. Given that the result does not depend on the precise definition of the domains one usually does not bother defining them and simply speaks of a sum over *regions* which are defined by their expansions alone. Common scenarios that involve more than two regions are:

- Hard scattering of energetic particles with $n_i = p_i/E_i$, $\bar{n}_i$ s.t. $n_i \cdot \bar{n}_i = 2$ and $\lambda \ll 1$:

$$\begin{array}{ll} \text{hard:} & l \sim Q \\ n_i\text{-collinear:} & (n_i \cdot l, \bar{n}_i \cdot l, l_\perp) \sim Q(\lambda^2, 1, \lambda) \\ \text{soft:} & l \sim Q\lambda \\ \text{ultrasoft:} & l \sim Q\lambda^2 \end{array} \tag{7.16}$$

In addition to the above some situations also require:

$$\begin{array}{ll} n_i\text{-coft:} & (n_i \cdot l, \bar{n}_i \cdot l, l_\perp) \sim Q\beta(\lambda^2, 1, \lambda) \\ \text{Glauber:} & (n \cdot l, \bar{n} \cdot l, l_\perp) \sim Q(\lambda^a, \lambda^b, \lambda) \text{ with } (a,b) = (2,2), (2,1), (1,2) \end{array} \tag{7.17}$$

- Non-relativistic QCD for heavy quarkonium with velocity $v \ll 1$:

$$\begin{array}{lll} \text{hard:} & l^0 \sim m & \mathbf{l} \sim m \\ \text{soft:} & l^0 \sim mv & \mathbf{l} \sim mv \\ \text{potential:} & l^0 \sim mv^2 & \mathbf{l} \sim mv \\ \text{ultrasoft:} & l^0 \sim mv^2 & \mathbf{l} \sim mv^2 \end{array} \tag{7.18}$$

Let us list some important properties of the expansion by regions:

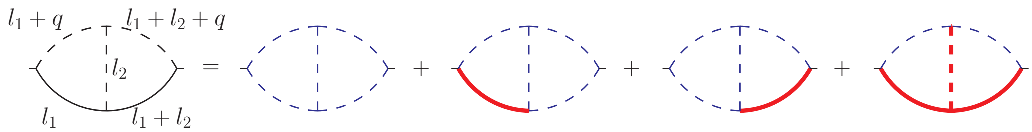
- One can always add irrelevant regions since they only lead to scaleless integrals which vanish. Sometimes regions which only yield scaleless integrals are required to achieve a decomposition of the integration domain [51]. They are typically omitted in practical applications.

- The overlap contributions vanish if the regions (and regulators) are chosen such that the contribution $F_j^{(x)}$ from each region is a homogeneous function of the expansion parameter with unique scaling. In our example this is the case with $F_j^{(h)} \sim (m^2)^j$ and $F_j^{(s)} \sim (m^2)^{j-\epsilon}$. It also holds in the vast majority of applications, at least when the region expansions are strictly applied. One well-known exception is SCET in the label-formalism where momentum conservation is enforced instead of properly expanding out small momentum components according to the scalings (7.16). In this context the overlap contributions are called zero-bin subtractions and have to be taken into account, see the discussion in [52].

- Even when the original integrals are well-defined in dimensional regularization, the expansion by regions requires additional regularization in some scenarios. An example considered below is the collinear anomaly [53, 52].

- At the Lagrangian level one can split the fields into momentum modes which correspond to the regions. Then one can construct effective theories by integrating out one or more of the momentum modes and expanding the result according to the power counting. The Feynman rules in the effective theories directly generate the expanded integrals $F_j^{(x)}$.

- In general it is not sufficient to consider a fixed assignment of the momenta and sum all loop momenta over all regions. As an example consider the 2-loop massive propagator integral again for $q^2 \gg m^2$



  where massive lines are solid and massless ones dashed. On the right hand side the contributions in the expansion by regions are shown where thick red lines are soft and thin blue lines hard. The left and right external vertex must be connected by a hard subgraph to allow the large momentum q to flow through the diagram and contributions with soft massless tadpoles (after all hard lines have been contracted to points) are not shown because they are scaleless. With the indicated momentum assignment the third contribution would be missed if one naively takes the sum over l_1 and l_2 being either hard or soft. Instead one should consider the sum over all possible assignments of regions to subgraphs.

- The expansion by region can also be applied to other types of integrals, e.g. Feynman parameter integrals. This is e.g. discussed in [1].

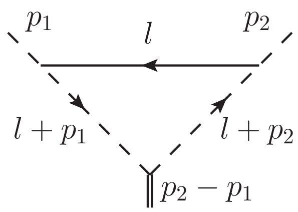


Figure 13: Massive one-loop correction to the Sudakov form factor.

A more complicated example involving more than two regions and the necessity of additional regularization is the massive Sudakov problem [51, 52] shown in Figure 13

$$F = \int \frac{[dl]}{[(l+p_1)^2]^{n_1}[(l+p_2)^2]^{n_2}[l^2-m^2]^{n_3}} \equiv \int [dl] I\,, \tag{7.19}$$

with $p_1^2 = p_2^2 = 0$ and $(p_1-p_2)^2 = -Q^2 < 0$ which we want to determine as an expansion in m^2/Q^2. We expect that the regions are given by (7.16) with $\lambda^2 = m^2/Q^2$ since we have scattering of energetic particles. We define the light-like reference vectors as $n^\mu = 2p_1^\mu/\sqrt{Q^2}$ and $\bar{n}^\mu = 2p_2^\mu/\sqrt{Q^2}$ which satisfy $n\cdot\bar{n} = 2$. This way we can decompose a given momentum as

$$l^\mu = \frac{\bar{n}\cdot l}{2} n^\mu + \frac{n\cdot l}{2}\bar{n}^\mu + l_\perp\,, \tag{7.20}$$

where $n\cdot l_\perp = \bar{n}\cdot l_\perp = 0$ and scalar products take the form

$$l\cdot k = \frac{(\bar{n}\cdot l)(n\cdot k) + (\bar{n}\cdot k)(n\cdot l)}{2} + l_\perp\cdot k_\perp\,, \qquad l^2 = (\bar{n}\cdot l)(n\cdot l) + l_\perp^2\,. \tag{7.21}$$

The regions (7.16) have the corresponding expansions

$$T^{(h)} I = \sum_{j=0}^{\infty} \frac{(n_3)_j}{j!} \frac{(m^2)^j}{[(l+p_1)^2]^{n_1}[(l+p_2)^2]^{n_2}[l^2]^{n_3+j}}\,, \tag{7.22}$$

$$T^{(1c)} I = \sum_{j=0}^{\infty} \frac{(n_2)_j}{j!} \frac{(-l^2)^j}{[(l+p_1)^2]^{n_1}[2l\cdot p_2]^{n_2+j}[l^2-m^2]^{n_3}}\,, \tag{7.23}$$

$$T^{(2c)} I = \sum_{j=0}^{\infty} \frac{(n_1)_j}{j!} \frac{(-l^2)^j}{[2l\cdot p_1]^{n_1+j}[(l+p_2)^2]^{n_2}[l^2-m^2]^{n_3}}\,, \tag{7.24}$$

$$T^{(s)} I = \sum_{j_1,j_2=0}^{\infty} \frac{(n_1)_{j_1}(n_2)_{j_2}}{j_1!j_2!} \frac{(-l^2)^{j_1+j_2}}{[2l\cdot p_1]^{n_1+j_1}[2l\cdot p_2]^{n_2+j_2}[l^2-m^2]^{n_3}}\,, \tag{7.25}$$

$$T^{(us)} I = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(n_1)_{j_1}(n_2)_{j_2}(n_3)_{j_3}}{j_1!j_2!j_3!} \frac{(-l^2)^{j_1+j_2+j_3}}{[2l\cdot p_1]^{n_1+j_1}[2l\cdot p_2]^{n_2+j_2}[-m^2]^{n_3+j_3}}\,, \tag{7.26}$$

where $(1c)$ and $(2c)$ are the regions collinear to $n \sim p_1$ and $\bar{n} \sim p_2$, respectively. The soft and ultrasoft regions do not contribute because the integrals

$$\begin{aligned}
&\int \frac{[dl]}{[2l\cdot p_1]^{\lambda_1}[2l\cdot p_2]^{\lambda_2}[l^2-m^2]^{\lambda_3}} \\
=&\frac{\Gamma(\lambda_{123})}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}\int_0^\infty dx_1\int_0^\infty dx_2\int\frac{[dl]x_1^{\lambda_1-1}x_2^{\lambda_2-1}}{[l^2+2l\cdot(x_1p_1+x_2p_2)-m^2]^{\lambda_{123}}} \\
=&(-1)^{n_{123}}\frac{\Gamma(\lambda_{123}-2+\epsilon)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}\int_0^\infty dx_1\int_0^\infty dx_2\frac{x_1^{\lambda_1-1}x_2^{\lambda_2-1}}{[x_1x_2Q^2+m^2]^{\lambda_{123}-2+\epsilon}} \\
=&(-1)^{n_{123}}\frac{\Gamma(\lambda_{13}-2+\epsilon)}{\Gamma(\lambda_1)\Gamma(\lambda_3)}(m^2)^{2-\lambda_{13}-\epsilon}(Q^2)^{-\lambda_2}\int_0^\infty dx_1x_1^{\lambda_1-\lambda_2-1}=0\,, \qquad (7.27)
\end{aligned}$$

are scaleless. Here, we already see the necessity for an analytic regulator because the integrals are ill-defined if the exponent $\lambda_1-\lambda_2-1$ is integer-valued. By analyzing the scaling of the integrands we see that the contributions of the remaining regions are homogeneous

$$\begin{aligned}
F_j^{(h)} &\sim (Q^2)^{2-\epsilon-n_{123}-j}(m^2)^j\,, \\
F_j^{(c1)} &\sim (Q^2)^{-n_2-j}(m^2)^{2-\epsilon-n_{13}+j}\,, \\
F_j^{(c2)} &\sim (Q^2)^{-n_1-j}(m^2)^{2-\epsilon-n_{23}+j}\,. \qquad (7.28)
\end{aligned}$$

For $n_i=1$ the two collinear regions however exhibit the same scaling and the vanishing of overlap contributions is not guaranteed. We see that it is sufficient to use an analytic regulator $n_1=1+\alpha$ or $n_2=1+\alpha$ to obtain unique scaling of all regions which is done in[52]. For now we keep all $n_i\neq 1$ and obtain

$$F=F^{(h)}+F^{(c1)}+F^{(c2)}\,. \qquad (7.29)$$

The result for the hard region follows from a standard calculation using Feynman parameters

$$\begin{aligned}
F_j^{(h)} =& \frac{(n_3)_j}{j!}\frac{(-m^2)^j}{(Q^2)^{n_{123}+j-2+\epsilon}}e^{-i\pi n_{123}} \\
&\times\frac{\Gamma(n_{123}+j-2+\epsilon)\Gamma(2-\epsilon-n_{13}-j)\Gamma(2-\epsilon-n_{23}-j)}{\Gamma(n_1)\Gamma(n_2)\Gamma(4-2\epsilon-n_{123}-j)}\,. \qquad (7.30)
\end{aligned}$$

The 1−collinear contribution is

$$\begin{aligned}
F_j^{(c1)} =&\frac{(n_2)_j}{j!}\int\frac{[dl][-(l^2-m^2)-m^2]^j}{[(l+p_1)^2]^{n_1}[2l\cdot p_2]^{n_2+j}[l^2-m^2]^{n_3}} \\
=&\frac{(n_2)_j}{j!}(-1)^j\sum_{i=0}^{j}\binom{j}{i}(m^2)^{j-i}\int\frac{[dl]}{[(l+p_1)^2]^{n_1}[2l\cdot p_2]^{n_2+j}[l^2-m^2]^{n_3-i}}, \qquad (7.31)
\end{aligned}$$

where the integral with arbitrary indices λ_i yields

$$\begin{aligned}
&\frac{\Gamma(\lambda_{123})}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)} \int_0^\infty \left[\prod_{i=1}^{3} dx_i x_i^{\lambda_i - 1}\right] \int \frac{[dl]\delta(1-x_{13})}{[x_{13}l^2 + 2l\cdot(x_1 p_1 + x_2 p_2) - x_3 m^2]^{\lambda_{123}}} \\
=&e^{-i\pi\lambda_{123}} \frac{\Gamma(\lambda_{123}-2+\epsilon)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)} \int_0^1 dx_1 \int_0^\infty dx_2 \frac{x_1^{\lambda_1-1}(1-x_1)^{\lambda_3-1} x_2^{\lambda_2-1}}{[x_1 x_2 Q^2 + (1-x_1)m^2]^{\lambda_{123}-2+\epsilon}} \\
=&e^{-i\pi\lambda_{123}} \frac{\Gamma(\lambda_{13}-2+\epsilon)}{\Gamma(\lambda_1)\Gamma(\lambda_3)} \frac{(m^2)^{2-\epsilon-\lambda_{13}}}{(Q^2)^{\lambda_2}} \int_0^1 dx_1 x_1^{\lambda_1-\lambda_2-1}(1-x_1)^{1-\lambda_1-\epsilon} \\
=&e^{-i\pi\lambda_{123}} \frac{\Gamma(\lambda_{13}-2+\epsilon)\Gamma(\lambda_1-\lambda_2)\Gamma(2-\lambda_1-\epsilon)}{\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(2-\lambda_2-\epsilon)} \frac{(m^2)^{2-\epsilon-\lambda_{13}}}{(Q^2)^{\lambda_2}} \,.
\end{aligned} \tag{7.32}$$

This is an example for an integral where dimensional regularization is not sufficient as $\Gamma(\lambda_1 - \lambda_2)$ diverges for integer λ_i with $\lambda_2 \geq \lambda_1$. This is an artifact of the expansion by regions. The original integral (7.19) is well-defined in dimensional regularization for arbitrary propagator powers. We obtain

$$\begin{aligned}
F_j^{(c1)} =&e^{-i\pi n_{123}} \frac{(n_2)_j}{j!} \frac{(m^2)^{2-\epsilon-n_{13}+j}}{(Q^2)^{n_2+j}} \frac{\Gamma(n_1-n_2-j)\Gamma(2-n_1-\epsilon)}{\Gamma(n_1)\Gamma(2-n_2-j-\epsilon)} \\
&\times \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\Gamma(n_{13}-i-2+\epsilon)}{\Gamma(n_3-i)} \\
=&e^{-i\pi n_{123}} \frac{(n_2)_j}{j!} \frac{(-1)^j (m^2)^{2-\epsilon-n_{13}+j}}{(Q^2)^{n_2+j}} \\
&\times \frac{\Gamma(n_1-n_2-j)\Gamma(2-n_1+j-\epsilon)\Gamma(n_{13}-j-2+\epsilon)}{\Gamma(n_1)\Gamma(n_3)\Gamma(2-n_2-j-\epsilon)} \,,
\end{aligned} \tag{7.33}$$

where the sum has been obtained as follows

$$\begin{aligned}
&\sum_{i=0}^{j} \binom{j}{i} (-1)^i \frac{\Gamma(n_{13}-2+\epsilon-i)}{\Gamma(n_3-i)} \\
=&\frac{1}{\Gamma(2-n_1-\epsilon)} \int_0^1 dx\, x^{n_{13}-3+\epsilon} (1-x)^{1-n_1-\epsilon} \underbrace{\sum_{i=0}^{j} \binom{j}{i} \left(-\frac{1}{x}\right)^i}_{(-1)^j x^{-j} (1-x)^j} \\
=&(-1)^j \frac{\Gamma(n_{13}-2+\epsilon-j)\,\Gamma(2-n_1-\epsilon+j)}{\Gamma(2-n_1-\epsilon)\,\Gamma(n_3)} \,.
\end{aligned} \tag{7.34}$$

The 2−collinear region follows from $F_j^{(c1)}$ under the exchange $n_1 \leftrightarrow n_2$. Being interested in the full result for $n_i = 1$ we can now take this limit while keeping ϵ finite. The results for the individual regions depend on how the limit is taken but the sum must obviously be independent. Here we choose $n_1 = 1+\delta$, $n_2 = 1-\delta$ and $n_3 = 1$ and obtain

$$F^{(h)} = \frac{e^{-\epsilon\gamma_E}}{(Q^2)^{1+\epsilon}} \left[\frac{-1}{\epsilon^2} + \frac{\pi^2}{12} + \left(\frac{-2}{\epsilon} + 2\right)\frac{m^2}{Q^2} + \left(\frac{-1}{\epsilon} - \frac{1}{2}\right)\left(\frac{m^2}{Q^2}\right)^2 + \dots\right] ,$$

$$F^{(c1)} = \frac{e^{-\epsilon\gamma_E}}{(Q^2)^{1-\delta}(m^2)^{\delta+\epsilon}} \left[\frac{-1}{\epsilon\delta} + \frac{1}{2\epsilon^2} - \frac{5\pi^2}{24} + \left(\frac{-1}{2\delta} + \frac{1}{\epsilon} - \frac{1}{2} \right) \frac{m^2}{Q^2} \right.$$
$$\left. + \left(\frac{-1}{4\delta} + \frac{1}{2\epsilon} + \frac{3}{8} \right) \left(\frac{m^2}{Q^2} \right)^2 + \dots \right],$$
$$F^{(c2)} = \frac{e^{-\epsilon\gamma_E}}{(Q^2)^{1+\delta}(m^2)^{-\delta+\epsilon}} \left[\frac{1}{\epsilon\delta} + \frac{1}{2\epsilon^2} - \frac{5\pi^2}{24} + \left(\frac{1}{2\delta} + \frac{1}{\epsilon} - \frac{1}{2} \right) \frac{m^2}{Q^2} \right.$$
$$\left. + \left(\frac{1}{4\delta} + \frac{1}{2\epsilon} + \frac{3}{8} \right) \left(\frac{m^2}{Q^2} \right)^2 + \dots \right]. \tag{7.35}$$

Summing up the regions the dependence on δ cancels and we obtain a finite result

$$F\left(\vec{1}\right) = \frac{1}{Q^2} \left[-\frac{1}{2}\ln^2(z) - \frac{\pi^2}{3} + (1 - \ln(z))\, z + \left(\frac{1}{4} - \frac{1}{2}\ln(z) \right) z^2 + \dots \right], \tag{7.36}$$

where $z = m^2/Q^2$, in agreement with the expansion of the full result

$$F\left(\vec{1}\right) = -\frac{1}{Q^2} \left[\frac{1}{2}\ln^2(z) - \ln(z)\ln(1-z) - \mathrm{Li}_2(z) + \frac{\pi^2}{3} \right]. \tag{7.37}$$

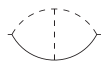
Eq. (7.37) can also be reproduced by summing the regions (7.30) and (7.33) to all orders [51], but typically we truncate the series expansion at some finite order.

It should be noted that we did not bother to define the integration domains for the regions or even check that they decompose the complete domain. This is hardly ever done in the literature and one often relies on the presence of uncanceled spurious divergences when a region is missed. However, not all regions give such divergences and especially generic off-shell regions like the Glauber (7.17) or potential (7.18) modes require caution. In this example it turns out that two more regions are needed to cover the complete domain [51]

$$\begin{array}{ll} \text{Glauber:} & (n \cdot l, \bar{n} \cdot l, l_\perp) \sim Q(\lambda^2, \lambda^2, \lambda) \\ \text{collinear plane:} & (n \cdot l, \bar{n} \cdot l, l_\perp) \sim Q(1, 1, \lambda) \end{array} \tag{7.38}$$

They only yield scaleless integrals, but there is the additional complication that their expansions do not commute. However, the correctness of (7.29) has been demonstrated in [51] using a generalization of (7.15) to the case of non-commuting expansions.

7.1 Exercises

1. Determine the first two terms in the m^2/q^2 expansion of the two-loop massive propagator diagram  with indices equal to one.

   Hint: The regions were already shown above. We already considered the IBP reduction of the massless two-loop propagator diagram to factorized or recursively one-loop diagrams in Section 5.1. The results of Section 2.5 can be used for the soft-soft region.

Files: `Examples/EBR_Sudakov.m`, `Exercises/EBR_TwoLoopProp.m`

8 Summary and further reading

We have introduced a number of techniques for the computation of Feynman integrals. The focus has been to illustrate the main ideas of the approaches and to explain their application using examples. More general discussions of the various methods can be found in the literature cited in the respective sections. Some additional techniques and many additional examples are available in the book [1]. In the Feynman integral community there is a large and growing number of public codes which are being used actively in state-of-the art calculations. We have referenced those which are available at the time of writing in the text and strongly encourage their use.

A priori it is often not obvious which particular method should be used for a given Feynman integral. Many problems also require the combination of different techniques. In recent years the method of differential equations has been very successful in calculating many master integrals for the first time. To set up the differential equations one first has to work out the IBP reduction for the integral family. The determination of the required boundary values is one example where other techniques can be applied. E.g. we saw in the Bhabha scattering example how the boundary condition can be determined by expanding the considered integral in a kinematic limit with the Mellin Barnes technique. The same result can also be obtained using the expansion by regions which is typically preferable for more complicated cases where very high-dimensional Mellin Barnes representation would be required. Sector decomposition on the other hand is very useful for numerical checks of analytical results for master integrals, but has also been used to perform entire calculations numerically.

Last but not least we give some references for related topics which are beyond the scope of these lectures:

- Generation of Feynman diagrams: `QGRAF` [54] and `FeynArts` [56]
- Symbolic manipulation system fur huge expressions: `FORM` [55]
- Codes for one-loop calculations: `FeynCalc` [57], `FormCalc` [58], `Package-X` [59]
- Analytic results for all QCD one-loop master integrals: [60] and references therein
- Databank of known master integrals: `Loopedia` [61]
- Review of one-loop techniques including tensor reduction [62]
- Review of electroweak corrections and renormalization [63]
- COmpendium of RElations: [64]

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