- Physical implications

5d parameters : M, k, r_c^{-1} (or Λ)

low energies $E \ll M, k, r_c \Rightarrow$ eff 4d theory

massless fluctuations : 4d graviton $h_{\mu\nu}(x)$ + radion $R(x)$

$$ds^2 = \left\{ e^{-2kR(x)|\phi|}\, \eta_{\mu\nu} + \bar{h}_{\mu\nu}(x) \right\} dx^\mu dx^\nu + R^2(x)\, d\phi^2$$

$$\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \bar{h}_{\mu\nu}(x) \qquad R(x) = r_c + r(x)$$

* no KK photon $A_\mu dx^\mu d\phi$ [no 5d isometries $\Rightarrow A_\mu$ massive] due to the presence of branes

* assume $R(x) = r_c$ fixed with $r(x)$ massive

$\Rightarrow$(1) 4d Planck mass : $\int d^4x \int_{-\pi}^{\pi} d\phi \; 2M^3 \underbrace{\sqrt{-G}\, R^{(5)}}$

$\hookrightarrow r_c \left(\sqrt{g}\, R^{(4)}\right)\left[e^{-2kr_c|\phi|} g_{\mu\nu}\right]$

$= r_c\, e^{-2kr_c|\phi|} \sqrt{g}\, R^{(4)} [\bar{g}_{\mu\nu}]$

$\left[\sqrt{g} R \sim g g \cdot g^{-1} g^{-1} \partial\partial g \sim g \right]$ ← scaling

$\equiv 2M_p^2 \int d^4x \sqrt{\bar{g}}\, \bar{R}^{(4)}$

$$\Rightarrow M_p^2 = M^3 r_c \int_{-\pi}^{\pi} d\phi\, e^{-2kr_c|\phi|} = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right)$$

$$\simeq M^3/k \quad (\text{for } kr_c \gg 1)$$

like one ~~extra~~ extra flat dim of radius k^{-1}

(2) Coupling of 3-brane fields to low energy gravitational fields

Hid : $g_{hid} = \bar{g}$ Vis : $g_{vis} = e^{-2kr_c\pi}\, \bar{g}$

↑ exp mass hierarchy

e.g. Higgs field

$$S_{vis} \supset \int d^4x \sqrt{-g_{vis}} \left\{ g^{\mu\nu}_{vis} D_\mu H^+ D_\nu H - \lambda \left(|H|^2 - v_0^2 \right)^2 \right\}$$

$$= \int d^4x \sqrt{-\bar{g}}\, e^{-4kr_c\pi} \left\{ \bar{g}^{\mu\nu} e^{2kr_c\pi} D_\mu H^+ D_\nu H - \lambda \left(|H|^2 - v_0^2 \right)^2 \right\}$$

wave function ~~r~~enormalization for canonical kinetic terms:

$$H \to e^{kr_c\pi} H \Rightarrow$$

$$\int d^4x \sqrt{-\bar{g}} \left\{ \bar{g}^{\mu\nu} D_\mu H^+ D_\nu H - \lambda \left(|H|^2 - e^{-2kr_c\pi} v_0^2 \right)^2 \right\}$$

$$\Rightarrow v = e^{-kr_c\pi} v_0 \longrightarrow \underbrace{m}_{\text{4d mass on vis brane}} = e^{-kr_c\pi} \underbrace{m_0}_{\text{5d mass}}$$

5d masses $\sim M_P \big|_{\text{reduced}} \sim 2 \times 10^{18}$ GeV

need $e^{kr_c\pi} \sim 10^{15} \longrightarrow$ TeV scale $\Rightarrow kr_c \simeq 30$

* Alternative picture on vis brane fund scale is TeV

but graviton localized in UV-brane (far) $\Rightarrow$

exp small overlap of wave function $\rightarrow$ weak 4d grav

but all other couplings (e.g. KK spin-2 excitations)

are expected $\sim \mathcal{O}(\text{TeV})$

RS2

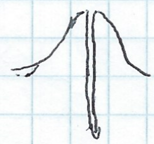
$$ds^2 = e^{-2k|y|}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + dy^2 \qquad r_c \to \infty$$

$$h(x,y) = e^{ipx}\,\psi(y) \qquad p^2 = m^2 \quad \Longrightarrow \quad \Big[\underbrace{-\tfrac{1}{2}\partial_y^2 - \tfrac{m^2}{2}e^{2k|y|} - 2k\,\delta(y)}_{\text{wave operator}} + \underbrace{2k^2}_{\text{curvature effect}}\Big]\psi(y) = 0$$

$$z = \operatorname{sgn} y\left(e^{k|y|} - 1\right)\frac{1}{k}$$

$$\Rightarrow \left[-\tfrac{1}{2}\partial_z^2 + V(z)\right]\hat{\psi}(z) = \frac{m^2}{2}\hat{\psi}(z) \qquad \hat{\psi}(z) = \psi(y)\,e^{k|y|/2}$$

$$\text{with } V(z) = \frac{15k^2}{8\left(k|z|+1\right)^2} - \frac{3k}{2}\delta(z)$$



$\Rightarrow$ 1 bound state + continuum (normalizable zero mode)

$$\frac{dz}{dy} = \begin{cases} kz+1 & y>0 \quad \left(z = \frac{e^{ky}-1}{k} \Rightarrow \frac{dz}{dy} = e^{ky}\right) \\ 1-kz & y<0 \quad \left(z = -\frac{e^{-ky}-1}{k} \Rightarrow \frac{dz}{dy} = e^{-ky} = 1-kz\right) \end{cases}$$

- $$\hat{\psi}_0(z) = \frac{1}{k}\,\frac{1}{\left(k|z|+1\right)^{3/2}}$$

- continuum: $$\left(|z| + 1/k\right)^{1/2}\Big\{\underbrace{Y_2\left(m\left(|z|+1/k\right)\right) + J_2\left(m\left(|z|+1/k\right)\right)}_{\text{linear combination}}\Big\}$$ Bessel functions

$$Y_2(x) \underset{x\to 0}{\sim} -\frac{4}{\pi x^2} - \frac{1}{\pi}$$

$$J_2(x) \underset{x\to 0}{\sim} \frac{x^2}{8} \qquad x \equiv m\left(|z| + 1/k\right)$$

boundary conditions due to δ-function $\Rightarrow$ $$\hat{\psi}_m \sim N_m\, \underbrace{x^{1/2}}_{\text{without } m}\left[Y_2(x) + \frac{4k^2}{\pi m^2}J_2(x)\right]$$

$$\sqrt{z}\,J_2(mz) \underset{mz\to\infty}{\sim} \sqrt{\frac{2}{\pi m}}\cos\left(mz - \frac{5\pi}{4}\right)$$

$$\sqrt{z}\,Y_2(mz) \sim \sqrt{\frac{2}{\pi m}}\sin\left(mz - \frac{5}{4}\pi\right)$$

$$\partial_z\hat{\psi}(0) = -\frac{3k}{2}\hat{\psi}(0) \qquad \cancel{\left(\hat{\psi}(0) \sim \underbrace{N_m}_{m^{1/2}/k^2}\,\frac{-4}{\pi}\,\frac{k^{3/2}}{m^2}\right)}$$

$$z = \operatorname{sgn} y \left(e^{k|y|} - 1\right)\frac{1}{k} = \pm\left(e^{\pm yk} - 1\right)\frac{1}{k}$$ *

($z>0$, $y<0$)

$$\frac{\partial z}{\partial y} = e^{k|y|}$$

$$\Rightarrow \left.\frac{\partial z}{\partial y}\right|_0 = 1 \qquad \left.\frac{\partial z}{\partial y}\right|_{y_c} = e^{ky_c} = kz_c + 1$$

$$\Rightarrow \delta(y - y_0) = \left.\frac{\partial y}{\partial z}\right|_{y_0} \delta(z - z_0) \Rightarrow \delta(y) = \delta(z)$$

$$\delta(y_c) = \frac{1}{kz_c + 1}\,\delta(z - z_c)$$

Boundary conditions:

$z{=}0$ $$0 = \int_{-\varepsilon}^{\varepsilon} -\frac{1}{2}\partial_z^2 \hat\varphi - \frac{3k}{2}\int \delta(z)\,\hat\varphi(z) =$$

$$= -\frac{1}{2}\left(\partial_z\hat\varphi\big|_{\varepsilon} - \partial_z\hat\varphi\big|_{-\varepsilon}\right) - \frac{3}{2}k\,\hat\varphi(0)$$

$$= -\,\partial_z\hat\varphi(0) - \frac{3}{2}k\,\hat\varphi(0)$$

$z = z_c$ $$\delta(y - y_0) = \frac{1}{kz_c + 1}\,\delta(z - z_c) \Rightarrow \partial_z\hat\varphi\big|_{z_c} = -\frac{3k}{2(kz_c + 1)}\,\hat\varphi(z_c)$$

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*

- $y > 0 \Rightarrow kz = e^{ky} - 1 > 0$
- $y < 0 \Rightarrow -kz = e^{-ky} - 1 < 0$

$$\left.\right\} \Rightarrow e^{k|y|} = k|z| + 1$$

$$\partial_z^2 \left[e^{k|y|/2} \psi(y) \right] = \partial_z \left\{ \underbrace{\frac{\partial y}{\partial z}}_{e^{-k|y|}} \underbrace{\partial_y \left(e^{k|y|/2} \psi \right)}_{e^{k|y|/2}\left(\partial_y \psi + \frac{k}{2} \operatorname{sgn} y \, \psi\right)} \right\}$$

$$= \partial_z \left\{ e^{-k|y|/2} \left(\partial_y \psi + \tfrac{k}{2} \operatorname{sgn} y \, \psi \right) \right\}$$

$$= \underbrace{\frac{\partial y}{\partial z}}_{e^{-k|y|}} \underbrace{\partial_y \left\{ e^{-k|y|/2} \left(\partial_y \psi + \tfrac{k}{2} \operatorname{sgn} y \, \psi \right) \right\}}_{e^{-k|y|/2}\left[\partial_y^2 \psi + \cancel{\frac{k}{2} \operatorname{sgn} y \, \partial_y \psi} - \frac{k}{2} \operatorname{sgn} y \left(\cancel{\partial_y \psi} + \frac{k}{2} \operatorname{sgn} y \, \psi \right) \right] + k \, \delta(y) \psi}$$

$$= e^{-3k|y|/2} \left[\partial_y^2 \psi - \frac{k^2}{4} \psi \right] + k \, \delta(y) \psi$$

$$\Rightarrow -\frac{1}{2} \partial_z^2 \hat{\psi} + \frac{15 k^2}{8 (k|z|+1)^2} \hat{\psi} = e^{-3k|y|/2} \left(-\frac{1}{2} \partial_y^2 \psi + \frac{k^2}{8} \psi \right) - \tfrac{k}{2} \delta(y) \psi$$

$$+ \frac{15 k^2}{8} e^{-2k|y|} e^{k|y|/2} \psi$$

$$= e^{-3k|y|/2} \left(-\frac{1}{2} \partial_y^2 \psi + 2 k^2 \psi \right) - \tfrac{k}{2} \delta(y) \psi = \frac{m^2}{2} \underbrace{e^{k|y|/2} \psi}_{e^{-3k|y|/2} \frac{m^2}{2} e^{2k|y|} \psi}$$

$y \neq 0, \pi$

Boundary condition on the wave function

$$\hat{\varphi}_m = N_m \, x^{1/2} \, \varphi_m \qquad \varphi_m = Y_2 + \alpha J_2(x)$$

$$x \to m\left(|z| + 1/k\right)$$

$$\partial_z \Rightarrow \underbrace{\frac{1}{2} m \, x^{-1/2}\Big|_{z=0}}_{m^{1/2} k^{1/2}} \varphi_m(0) + m \left(\frac{m}{k}\right)^{1/2} \varphi_m'(0) = -\frac{3}{2} \underbrace{k \left(\frac{m}{k}\right)^{1/2}}_{m^{1/2} k^{1/2}} \varphi_m(0)$$

$$\Rightarrow 2 k^{1/2} \varphi_m(0) + m k^{-1/2} \varphi_m'(0) = 0$$

$$\Rightarrow -2\left[Y_2(\omega) + \alpha J_2(\omega) \right] = \omega \left[Y_2' + \alpha J_2' \right] \qquad \omega = m/k$$

$$m \sim \omega \sim 0 \Rightarrow -2\left[\cancel{-\frac{4}{\pi\omega^2}} - \frac{1}{\pi} + \alpha \frac{\omega^2}{8} \right] = \omega \left[\cancel{\frac{8}{\pi \omega^2}} + \alpha \frac{\omega}{4} \right]$$

$$\Rightarrow \frac{2}{\pi} = \alpha \frac{\omega^2}{2} \Rightarrow \boxed{\alpha = \frac{4}{\pi \omega^2} = \frac{4k^2}{\pi m^2}}$$

$m \sim 0 \qquad N_m \sim \frac{m^2}{k^2} \frac{1}{\sqrt{[illegible]}}$

$$\hat{\varphi}_m \Big|_{z=0} \sim \frac{m^2}{k^2} \frac{k^2}{m^2} \left(\frac{m}{k}\right)^{1/2} = \left(\frac{m}{k}\right)^{1/2}$$

$$\left(\hat{\varphi}\Big|_{z=0} \right)^2 \sim \frac{m}{k}$$

- $y = \pi r_c$ finite $\Rightarrow$ new boundary condition at $z_c \equiv \frac{1}{k}\left(e^{k\pi r_c}-1\right)$

$$\partial_z \hat{\psi}\Big|_{z_c} = -\frac{3k}{2(kz_c+1)}\hat{\psi}(z_c)$$

$\Rightarrow$ quantization of m

- z_c large $\Rightarrow$ plane wave $\Rightarrow$ $m \sim 1/z_c$ (quantized)

$\tilde{N}_m \sim \pi\, m^{3/2} / 4k^2\sqrt{z_c}$ ← from $\times^{1/2}$

measure : $\sqrt{z_c}\, dm$

volume factor

(*) wave function at $z_c \sim \frac{1}{\sqrt{z_c}} \times$ w.f. at 0 $\Rightarrow$ coupling $z_c \sim e^{k r_c \pi}$ stronger

~~$1/z_c$ ... $z^{3/2}$~~

* Newton's law, at $z=0$

$$V(r) = G_N \frac{m_1 m_2}{r} + \int_0^\infty dm\, \frac{G_N}{k}\, \frac{m_1 m_2 e^{-mr}}{r}\, \frac{m}{k}$$

0 mode

continuum

(5d) $\frac{1}{M^3}$

w.f. suppression at the origin

$$\Rightarrow V(r) \approx G_N \frac{m_1 m_2}{r}\left(1 + \frac{1}{r^3 k^2}\right)$$

corrections

(*) z_c large $\Rightarrow$ $\hat{\psi}_m \sim \sqrt{z}\, N$ $\Rightarrow$ $\int dz\, |\hat{\psi}|^2 \sim \int^{z_c} dz\, N^2 \sim z_c N^2$

normalization

$\Rightarrow N \sim \frac{1}{\sqrt{z_c}}$ $\Rightarrow$ $\hat{\psi}_m \sim \frac{1}{\sqrt{z_c}\,k}$ $\propto$ $z_c \times \hat{\psi}_0(z_c)$ ← $\frac{1}{(\sqrt{z_c})^3}$

$\Rightarrow$ KK modes couple as $z_c \sim e^{kr_c}$ stronger than 0-mode

• Radion stabilization ~~Goldberger-Wise~~
Goldberger-Wise mechanism '99

→ add a bulk scalar field with boundary potential

$$S_b = \frac{1}{2}\int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{G} \left\{ G^{MN} \partial_M \Phi \partial_N \Phi - m^2 \Phi^2 \right\}$$

$\phi = 0$ $\quad S_{hid} = -\int d^4x \sqrt{-g_h}\, \lambda_h \left(\phi^2 - v_h^2\right)$

$\phi = \pi$ $\quad S_{vis} = -\int d^4x \sqrt{-g_v}\, \lambda_v \left(\phi^2 - v_v^2\right)$

approx: $\lambda_h, \lambda_v \to \infty$ so that $\Phi(0) = v_h \quad \Phi(\pi) = v_v$

→ boundary conditions for bulk solution

$$ds^2 = e^{-2k r_c |\phi|} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 d\phi^2$$

$$\left[\frac{1}{\sqrt{G}} \partial_\phi \sqrt{G} \partial_\phi - m^2 r_c^2\right] \Phi = 0 \qquad \sqrt{G} = e^{-4\sigma} \qquad \sigma = k r_c |\phi|$$

$$\Rightarrow \left[\partial_\phi^2 - 4k r_c \partial_\phi - m^2 r_c^2\right] \Phi = 0$$

$\Phi = e^{\omega\phi}$: $\omega^2 - 4k r_c \omega - m^2 r_c^2 = 0 \qquad \omega = 2k r_c \pm \sqrt{4k^2 r_c^2 + m^2 r_c^2}$
$= k r_c \left(2 \pm \sqrt{4 + m^2/k^2}\right)$

$$\Rightarrow \Phi = e^{2k r_c |\phi|} \left[A\, e^{\nu r_c k|\phi|} + B\, e^{-\nu r_c k|\phi|}\right] \qquad \nu = \sqrt{4 + m^2/k^2}$$

A, B : $\Phi(0) = v_h \quad \Phi(\pi) = v_v$

$\rightarrow V(r_c) = \frac{1}{2}\int_{-\pi}^{\pi} d\phi \left[(\partial\phi)^2 - m^2\phi^2\right] \rightarrow$ minimization $\frac{m}{k} \to 0$: $\nu = 2 + \varepsilon$, $\varepsilon = m^2/4k^2$

$k r_c$ large $\Rightarrow$ $k r_c = \frac{4}{\pi}\left(\frac{k}{m}\right)^2 \ln \frac{v_h}{v_v} \sim 10$

$$\Phi = A\, e^{(2+\nu)\sigma} + B\, e^{(2-\nu)\sigma}$$

$$\Phi' = \left[(2+\nu) A\, e^{(2+\nu)\sigma} + (2-\nu) B\, e^{(2-\nu)\sigma}\right] k r_c$$

$$\frac{1}{2}\int \left[(\partial\phi)^2 - m^2\phi^2\right] = \frac{1}{2}\int \partial(\phi\,\partial\phi) - \phi(\Box + m^2)\phi$$

$\sqrt{G}\, g^{\mu\nu}$

$$\frac{1}{2} e^{-4\sigma} \underbrace{\phi\,\phi'}_{} \Big|_0^{k r_c \pi} \qquad \sigma = k r_c |\phi|$$

$$2 \to \frac{1}{4}(\phi^2)'$$

$$\Phi^2 = A^2 e^{2(2+\nu)\sigma} + B^2 e^{2(2-\nu)\sigma} + 2AB\, e^{4\sigma}$$

$$e^{-4\sigma}\frac{1}{2}(\phi^2)' = \underset{k r_c}{}\left[(2+\nu) A^2 e^{2(2+\nu)\sigma} + (2-\nu) B^2 e^{2(2-\nu)\sigma} + 4AB\, e^{4\sigma}\right] e^{-4\sigma}\Big|_0^{k r_c \pi}$$

$$= \left[(2+\nu) A^2 e^{2\nu\sigma} + (2-\nu) B^2 e^{-2\nu\sigma} + \cancel{4AB}\right]_0^{k r_c \pi} \quad k r_c / r_c$$

$$V_{eff} = (2+\nu) A^2 \left(e^{2\nu k r_c \pi} - 1\right) + (2-\nu) B^2 \left(e^{-2\nu k r_c \pi} - 1\right)$$

$$\left.\begin{aligned} &A + B = v_h \\ &A\, e^{(2+\nu) k r_c \pi} + B\, e^{(2-\nu) k r_c \pi} = v_v \end{aligned}\right|$$

$$\nu = 2 + \varepsilon \qquad \varepsilon = m^2 / 4k^2$$

$$\Rightarrow \left.\begin{aligned} &A + B = v_h \\ &A\, e^{4 k r_c \pi} + B = v_v \end{aligned}\right\} \Rightarrow \begin{aligned} &A = (v_v - v_h)\, e^{-4 k r_c \pi} \\ &B = v_h - v_v\, e^{-4 k r_c \pi} \end{aligned}$$

$r_c k$ large

but $\varepsilon r_c k$ finite

$$V_{eff} \simeq 4 A^2 e^{4 k r_c \pi} + \varepsilon B^2$$

$$= 4 (v_v - v_h)^2 e^{-4 k r_c \pi} + \varepsilon \left(v_h - v_v e^{-4 k r_c \pi}\right)^2$$

$$\left.\begin{aligned} &A + B = v_h \\ &A e^{4kr_c\pi} + B e^{\epsilon k r_c \pi} = v_v \end{aligned}\right\} \Rightarrow$$

$$A\left(e^{4kr_c\pi} - \cancel{e^{\epsilon k r_c\pi}}\right) = v_v - v_h e^{\epsilon k r_c \pi} \Rightarrow A \approx e^{-4kr_c\pi}\left(v_v - v_h e^{\epsilon k r_c\pi}\right)$$

$$B\left(e^{4kr_c\pi} - \cancel{e^{\epsilon k r_c \pi}}\right) = v_h e^{4kr_c\pi} - v_v \Rightarrow B \approx v_h - v_v e^{-4kr_c\pi}$$

$$\Rightarrow V_{eff} \sim 4A^2 e^{4kr_c\pi} + \epsilon B^2$$

$$\simeq 4 e^{-4kr_c\pi}\left(v_v - v_h e^{\epsilon k r_c\pi}\right)^2$$

$\Rightarrow$ minimum w.r.t. r_c $\Rightarrow$ $e^{\epsilon k r_c \pi} = \frac{v_v}{v_h}$

$$\Rightarrow kr_c = \frac{1}{\epsilon\pi}\ln\frac{v_v}{v_h} = \frac{4}{\pi}\frac{k^2}{m^2}\ln\frac{v_v}{v_h} \sim 10$$

Can easily choose parameters