- Brane: hypersurface of p-spatial dimensions (p-brane) embedded in a higher dimensional bulk ($D = 4+n > p+1$)

Field theory: topological defect

String theory: D-branes defined by the ends of open strings

branes can carry "charge" (necessary for stability)

point particle ≡ 0-brane can be electric source for gauge field (1-form)

$$e_0 \int dx^\mu A_\mu$$

string ≡ 1-brane : for 2-form gauge potential (2-index tensor)

$$e_1 \int dx^\mu \wedge dx^\nu A_{\mu\nu}$$

⋮

p-brane ⟶ (p+1)-form gauge potential

$$e_p \int \underbrace{dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}} A^{(p+1)}_{\mu_1 \dots \mu_p}$$

$$\int d^{p+1}\xi \; \varepsilon^{\alpha_1 \dots \alpha_n} \partial_{\alpha_1} X^{\mu_1} \dots \partial_{\alpha_{p+1}} X^{\mu_{p+1}} A^{(p+1)}_{\mu_1 \dots \mu_{p+1}} \quad (*)$$

ξ^α brane world volume coordinates

after defined brane

new notation

$$\int d^{p+1}x \; \varepsilon^{\alpha_1 \dots \alpha_{p+1}} \partial_{\alpha_1} Y^{M_1} \dots \partial_{\alpha_{p+1}} Y^{M_{p+1}} A^{(p+1)}_{M_1 \dots M_{p+1}}$$

eg. D3-brane in 10 d: $ds^2 = f^{-1/2} dx_{\parallel}^2 + f^{1/2}(dr^2 + r^2 d\Omega_5^2)$

unit 5-sphere

$f = 1 + \frac{4\pi g \, l^4}{r^4}$: harmonic function $\Delta_6 \frac{1}{r^4} = \delta^6(x_\perp)$

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Two branes: gravity attractive force
A   repulsive " (dine em) } ⇒ they cancel

Brane analogy with an extremal charged black hole

Charged BH (Reissner-Nordström solution)

$$ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) dt^2 + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2$$

2-sphere of unit radius $(d\theta^2 + \sin^2\theta \, d\varphi^2)$

$r_s = 2GM$

$r_Q^2 = e^2 G$

$\Rightarrow$ two horizons at $r = r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right)$

$A_\mu = \left(-\frac{e}{r}, \vec{0}\right)$

extremal case: $r_+ = r_- \Rightarrow r_s = 2r_Q \Rightarrow \boxed{GM^2 = e^2}$

$$\Rightarrow ds^2 = -\left(1 - \frac{r_Q}{r}\right)^2 dt^2 + \underbrace{\left(1 - \frac{r_Q}{r}\right)^{-2} dr^2 + r^2 d\Omega_2}_{f^2(R)\,[dR^2 + R^2 d\Omega_2]}$$

$r = R f(R) \Rightarrow \frac{dr}{1 - r_Q/r} = f(R)\, dR = \frac{r}{R} dR \Rightarrow \frac{dr}{r - r_Q} = \frac{dR}{R}$

$\Rightarrow r = r_Q + R$, $f(R) = \frac{r}{R} = 1 + \frac{r_Q}{R}$, $1 - \frac{r_Q}{r} = \frac{R}{r} = \frac{1}{f(R)}$

$$\Rightarrow ds^2 = -\frac{1}{f^2(R)} dt^2 + f^2(R)\left(dR^2 + R^2 d\Omega_2\right)$$

with $f(R) = 1 + \frac{r_Q}{R}$: harmonic function

$$\Delta_3 f(R) = r_Q \delta^3(x)$$

$\Rightarrow$ extremal BH $\equiv$ 0-brane

↳ point particle with charge $e = \sqrt{GM^2}$

force between two:

$$\frac{GM^2}{r} - \frac{e^2}{r} = 0 \quad \text{for } e^2 = \sqrt{GM^2}$$

gravity attractive

em - repulsive

no force $\Rightarrow$ stability

- branes with charge in the extremal limit:

  generalization of this situation

* Ex: 3-brane world

X^M ; $M=0,1,\dots,3+n$: spacetime coordinates of the bulk

x^μ ; $\mu=0,1,2,3$ coordinates of the 3-brane (our spacetime)

$Y^M(x)$: positions of the 3-brane in the extra dims

$(4+n)$-dim metric $G_{MN}(X)$

Bulk fields : $B(X)$

Brane-localized fields : $\Phi(x)$

Vacuum state: $G_{MN} = \eta_{MN}$ (flat space bulk)

$$Y^M = \delta^M_\mu x^\mu$$

→ describe fluctuations around the vacuum state

$$S_{bulk} = - \int d^{4+n}X \sqrt{|G|} \left\{ M_*^{2+n} R^{(4+n)} + \Lambda + \text{other bulk fields} \right\}$$

Induced metric on the brane:

$$ds^2 = \underbrace{G_{MN}}_{G_{MN}(Y(x))} dY^M(x)\, dY^N(x) = \underbrace{G_{MN} \frac{\partial Y^M}{\partial x^\mu} \frac{\partial Y^N}{\partial x^\nu}}_{g_{\mu\nu}\ \text{: induced metric}} dx^\mu dx^\nu$$

Symmetries of the Eff Action : general parametrization of the bulk

+ " " of the brane coordinates

⇒ ex. $\partial_\mu Y^M$: vector under bulk coordinate transformations

" " brane

$g_{\mu\nu}$: tensor under brane transfs

scalar " bulk

etc.

$$S_{brane} = -\int d^4x \sqrt{|g|} \left\{ T + R^{(4)} + \frac{1}{2} g^{\mu\nu} D_\mu \Phi D_\nu \Phi + \frac{1}{4} F^2_{\mu\nu} + \dots \right\}$$

gauge choice of brane coord transf. $x^\mu \to x'^\mu(x)$

4 conditions $\Rightarrow$ $Y^\mu(x) = x^\mu$, $Y^m(x)$ $m = 4, 5, \dots, 3+n$

brane positions in the n-extra dims $\nearrow$ physical degrees of freedom

$$G_{MN} = \eta_{MN} + \frac{1}{M_*^{1+n/2}} h_{MN} \leftarrow \text{higher dim graviton}$$

Brane fluctuations: $g_{\mu\nu} = G_{MN} \partial_\mu Y^N \partial_\nu Y^N = \eta_{\mu\nu} + \partial_\mu Y^m \partial_\nu Y_m + \dots$

$$\Rightarrow \det g = -1 - \partial_\mu Y^m \partial^\mu Y_m + \dots$$

T: brane tension ($T^{1/4} \ll M_*$ to neglect back reaction)

canonically normalized field: $Z^m = T^{1/2} Y^m$

For $Y = 0$ $\quad g_{\mu\nu} = G_{\mu\nu}(x^\lambda, x^m = 0)$ $\nwarrow$ brane position

$\Rightarrow$ For SM localized on the 3-brane $\quad S_{SM} = \int d^4x \, \mathcal{L}_{SM} \sqrt{g} (g_{\mu\nu}, \Phi, Z, \dots)$

$\Rightarrow$ Coupling of the brane fields to graviton:

$$S_{int} = \int d^4x \; T^{\mu\nu}_{SM} \frac{h_{\mu\nu}(x_\mu)}{M_*^{n/2+1}}$$

KK excitations: $\quad h_{\mu\nu}(x,y) = \sum_n e^{iny/R} \frac{h^{(n)}_{\mu\nu}}{\sqrt{2\pi R}}$

e.g. for 1 extra dim

$\nearrow$

in general $n \to \vec{n}$

$R \to \vec{R}$ for toroidal (or more general)

$\sqrt{2\pi R} \to \sqrt{V_{int}}$

$$\Rightarrow S_{int} = \int d^4x \; T^{\mu\nu}_{SM} \sum_k \underbrace{\frac{h^{k}_{\mu\nu}}{\sqrt{V_n}\; M_*^{n/2+1}}}_{4d \;\; M_P}$$

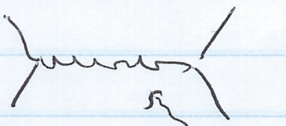
int volume

$$\left(M_P^2 = M_*^{2+n} \, V_n \right)$$

$\Rightarrow$ each KK mode couples with the same strength M_P (as the 0-mode)

$\rightarrow$ brane charge from p. 16

- Exchange of KK modes between brane sources



Ex of bulk gauge field or bulk graviton

propagator: $\dfrac{1}{k^2 + m_0^2 + \frac{n^2}{R^2}}$ $\times$ tensor structure

(one extra dim

m_0 : 5d mass

k : 4d momentum)

$$\sum_{n \in \mathbb{Z}} \frac{1}{a^2+n^2} = \frac{\pi}{a} \coth \pi a \Rightarrow$$

$$\sum_{\substack{KK\text{-modes} \\ \neq 0}} = \left(\frac{\pi}{Ra} \coth \pi R a - \frac{1}{R^2 a^2} \right) R^2 g^2 \qquad a = \sqrt{k^2+m_0^2}$$

4d coupling g

(can be $\frac{1}{M_P}$ for gravity)

$$\sum = g^2 \left(\frac{\pi R}{a} \coth \pi R a - \frac{1}{a^2} \right)$$

$$\coth x = \begin{cases} \pm 1 & x \to \pm\infty \\ \dfrac{1}{x} + \displaystyle\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \, x^{2n-1}}{(2n)!} = \frac{1}{x} + \frac{x}{3} + \dots & x \to 0 \end{cases}$$

Bernoulli numbers

limits:

- 4d limit $R \to 0 \Rightarrow \Sigma \to 0$
- $k = 0$ (0-momentum) $\Rightarrow a = m_0$

$m_0 = (k=0)$ $\Rightarrow \Sigma \to \frac{g^2}{3} \pi^2 R^2$: low energy eff operator of the form $(\text{current})^2$ ← $T_{\mu\nu}$ for gravity

- $m_0 = 0 \Rightarrow a = k \left(= \sqrt{k^2} \right)$
- $Rk \to \infty \Rightarrow \Sigma \sim g^2 \frac{\pi R}{k} \sim \frac{g_5^2}{2k}$ where $g^2 = \frac{g_5^2}{2\pi R}$

e.g. $\frac{1}{g_5^2} F^2 \to \frac{2\pi R}{g_5^2} \equiv \frac{1}{g_4^2}$ or $\underbrace{M_p^2}_{g_4^2} = \underbrace{M_*^3}_{g_5^2} 2\pi R$ for gravity

more dims: $2\pi R \to V_n$: volume of n-extra dims

- More than 1 extra dims $n \geq 2$: sum over $\vec{n}$ diverges

for $n=2$ logarithmic ; for $n \geq 3$ power Λ^{n-2}

$\Rightarrow$ need regularization: width S: $e^{-S\vec{n}^2}$