- Generalizations

- Non-abelian vectors A_M^a ← gauge group

$$A_M^a = \begin{pmatrix} A_\mu^a \\ \phi^a \equiv A_4^a \end{pmatrix} \Rightarrow \mathrm{Tr}\, F_{MN}^2$$

(ϕ^a — adjoint scalar)

$$\int dy\, \mathrm{Tr} F_{MN}^2 = \mathrm{Tr}\, F_{\mu\nu}^2(x) + \mathrm{Tr}(\partial\phi)^2 + \sum_{n\neq 0} (\text{massive vectors})$$

$$(\text{massive vectors}) = \mathrm{Tr}(F_{\mu\nu}^n)^2$$

$$F_{\mu\nu}^n = \partial_\mu A_\nu^n - \partial_\nu A_\mu^n + \sum_m [A_\mu^m, A_\nu^{n-m}]$$

- More extra dims $n > 1$ KK excitations (besides the 0-modes)

- $A_M \Rightarrow \begin{pmatrix} A_\mu \\ \phi^i \end{pmatrix}$ → massive vectors
→ $n-1$ massive scalars (adjoints)

- $G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu i} \\ G_{\nu j} & G_{ij} \end{pmatrix}$

0-modes: 4d graviton, n vectors, $\frac{n(n+1)}{2}$ scalars

KK-modes: spin-2 $G_{\mu\nu}^n$, $n-1$ vectors, $\frac{n(n-1)}{2}$ scalars

$$\frac{n(n-1)}{2} = \frac{n(n+1)}{2} - 1 - (n-1)$$

massive degrees of freedom: $5 + 3(n-1) + \frac{n(n-1)}{2} = 3n + 2 + \frac{n(n-1)}{2} = \frac{n^2+5n+4}{2}$

$$= \frac{(n+4)(n+1)}{2} = \frac{D(D-3)}{2}$$

↖ graviton in D dim

$$\frac{D(D+1)}{2} - D - D = \frac{D^2 - 3D}{2}$$

($-D$: gauge cond; $-D$: Gauss law)

Normalized fields: $\tilde{\Phi}_{\vec{n}} = \Phi_{\vec{n}} / \sqrt{V_{int}}$ ← internal volume

• Scalars G_{ij} : non-trivial compactification space
sizes and (complex structure) shape of the manifold

E.g. $n=2$ $\begin{pmatrix} G_{44} & G_{45} \\ G_{45} & G_{55} \end{pmatrix} \Rightarrow$ $\sqrt{G}$: radion
G_{45}/G_{55}, G_{44}/G_{55} : complex structure/shape

– Chirality and orbifolds

4d chirality from extra dims : non trivial

Higher dim spinor $\Rightarrow$ L,R symmetric in 4d

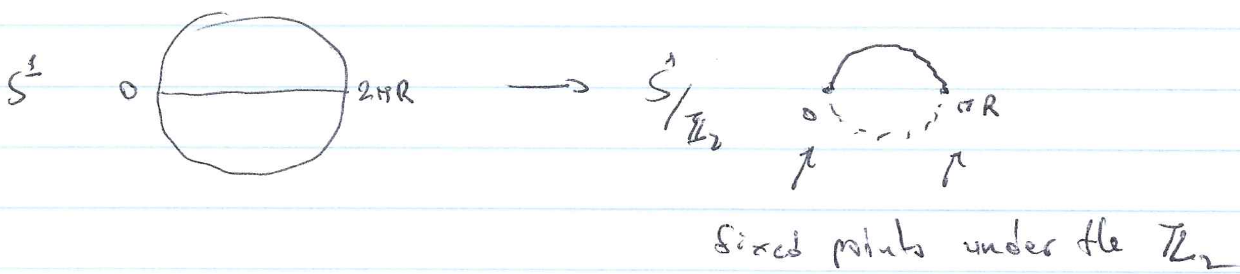
E.g. $D=5 \Rightarrow$ γ-matrices : $\{\gamma^\mu, \gamma^5\} = \gamma^M$

$\Rightarrow$ 5d spinor $\equiv$ 4d Dirac spinor (L + R component)

Need non-trivial manifold

Orbifolds : special limits allowing chirality
at the expense of singular (fixed) points

Ex: one dim on an interval $\equiv S^1/\mathbb{Z}_2$ ($y \equiv y + 2\pi R$; $y \to -y$)



Boundary conditions

- $\Phi(y + 2\pi R) \equiv \Phi(y) \Rightarrow \Phi(-y) = \pm \phi(y)$ — even (e) / odd (o)

$$\phi_{even} = \sum_n \cos \frac{n}{R} y \; \phi_n \qquad \frac{|n\rangle + |-n\rangle}{\sqrt{2}} \text{ KK}$$

invariant under $y \to -y$ and $y \to y + 2\pi R$

$$\phi_{odd} = \sum_n \sin \frac{ny}{R} \; \phi_n \qquad \frac{|n\rangle - |-n\rangle}{\sqrt{2}} \text{ KK}$$

ϕ_{odd}: no zero-modes $\quad \phi_{odd}(0) = \phi_o(\pi R) = 0 \,, \; \phi_o = 0 \quad n$: odd

ϕ_{even} : n: even $\quad \phi_{even}(0) = \phi_o = -\phi_{even}(\pi R)$

- Fermions

$$\psi(-y) = \gamma^4 \psi(y) \equiv \gamma^5 \psi(y) \Rightarrow \psi_L = \frac{1+\gamma_5}{2}\psi \text{ invariant}$$

$$\psi_R = \frac{1-\gamma_5}{2}\psi \to -\psi_R$$

$\Rightarrow$ 0-modes only for $\psi_L \Rightarrow$ 4d chirality

- Gauge fields

$$A_M(-y) = \begin{pmatrix} A_\mu(y) \\ -A_4(y) \end{pmatrix} \Rightarrow \begin{array}{l} A_\mu: \text{even} \\ A_4: \text{odd} \end{array}$$

$\Rightarrow A_\mu^{(n)}$ n even massive + 0-mode, no scalar

- metric

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu 4} \\ G_{4\nu} & G_{44} \end{pmatrix} \Rightarrow G_{\mu\nu}^{(n)} \quad n \text{ even}$$

($G_{\mu 4}$, $G_{4\nu}$: odd; G_{44}: even)

G_{44} = radion

no vectors

- fixed points : boundary states (twisted)

  fixed points : singular (breakdown of EoM)

  $\Rightarrow$ in general 4d anomalies extra states "localized" at the fixed points (boundaries)

  model dependent (twisted in string theory)

  $\Rightarrow$ notion of "branes"

- Coordinate dependent compactification, mass generation and symmetry breaking

- Scherk-Schwarz EQ

1) symmetry of higher dim theory $\Rightarrow$ general boundary conditions

   periodic up to symmetry transformation

   e.g. U(1) phase : $\phi(y+2\pi R) = e^{2i\pi\omega}\phi(y)$ ← complex field

   $$\Rightarrow \phi(y) = \sum_n e^{iy\frac{n+\omega}{R}}\phi_n$$

   $$S^1/\mathbb{Z}_2 \Rightarrow \phi_+ = \sum_n \cos y\frac{n+\omega}{R}\phi_n^+ \qquad \phi_- = \sum_n \sin y\frac{n+\omega}{R}\phi_n^-$$

   $\Rightarrow$ 0-mode gets a mass ω/R, $|n\rangle, |-n\rangle$ split (for an orbifold)

2) gauge symmetry breaking e.g. $SU(2) \to U(1)$ (T_3 generator)

   $$\Phi(y+2\pi R) = e^{-2i\pi\omega t_3}\,\Phi(y)\,e^{2i\pi\omega t_3} \qquad \Phi \equiv A_M$$

$\{\sigma^\pm, t_3\} = 0 \quad \Rightarrow \quad t_3 = \frac{\sigma_3}{2}$

$$\Phi_\pm(y+2\pi R) = e^{-4i\pi\omega t_3}\,\Phi_\pm(y) \quad , \quad \phi_3(y+2\pi R) = \phi_3(y)$$

$$= [\cos 2\pi\omega - i\sigma_3 \sin 2\pi\omega]\,\phi_\pm(y)$$

$\sigma^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2) = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \pm \frac{i}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$

$\Rightarrow \sigma^+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \quad \sigma^3\sigma^+ = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 1 \\ 0 & \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma^+$

$\sigma^3\sigma^- = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 0 \\ 1 & \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\sigma^-$

$\Rightarrow \phi_\pm(y+2\pi R) = e^{\mp 2i\pi\omega}\,\phi_\pm(y)$

$\Rightarrow \phi^\pm_n$ gets a mass^2 $(n \pm \omega)^2$

$\Rightarrow A^\pm_{\mu,n}$ absorbs $A^\pm_{4,n}$, $A^3_{\mu,0}$ massless , $A^3_{4,0}$ massless

$SU(2) \to U(1)$ by a vev of $\langle A_4 \rangle \sim \omega\sigma^3$

3) Supersymmetry breaking

use for $U(1)$ an R-symmetry

fermions (gravitino, gaugino) transform $\Rightarrow$ SUSY is broken

frequencies shifted by ω: $n + \omega \equiv m_n$

usually R-symmetry discrete $\Rightarrow$ ω is quantized

special case $\omega = \frac{1}{2}$ $\Rightarrow$ fermions antiperiodic $\Rightarrow$ ½-integer frequencies

as in finite temperature $y \to$ time