X-dimensions

unobservable $\rightarrow$ finite size

detection : KK spectrum

ex: 1 extra dim on a circle $\quad y \equiv y + 2\pi R$

$$\Rightarrow \phi(x^\mu, y) = \sum_{n\in\mathbb{Z}} \phi_n(x)\, e^{iyn/R}$$

$\Box_5 \phi = M^2 \phi \qquad$ signature : $(- + + \dots +)$

massive scalar field in 5d

$$\Box_5 = \Box_4 + \partial_y^2$$

$$\Rightarrow \left(\Box - \frac{n^2}{R^2}\right)\phi_n = M^2 \phi_n \quad \Rightarrow \quad \Box \phi_n = \Big(M^2 + \underbrace{\frac{n^2}{R^2}}_{\text{KK spectrum}}\Big)\phi_n$$

Effective action : $\displaystyle S = \frac{1}{2}\int d^4x \int_0^{2\pi R} dy \left\{ -(\partial_M \phi)^2 + M^2 \phi^2 \right\}$

$$\int dy\, e^{i(m+n)y} = 2\pi R\, \delta_{m+n,0}$$

$$\Rightarrow S = \sum_n \frac{1}{2}\int d^4x\; 2\pi R \Big\{ -(\partial_\mu \phi_n)^2 \underbrace{- \frac{nm}{R^2}\phi_n \phi_m \delta_{n+m}}_{\frac{n^2}{R^2}\phi_n^2} + M^2 \phi_n^2 \Big\}$$

$$= \frac{2\pi R}{2}\int d^4x \sum_n \left\{ -(\partial_\mu \phi_n)^2 + \left(M^2 + \frac{n^2}{R^2}\right)\phi_n^2 \right\}$$

normalized field: $\displaystyle \tilde{\phi}_n = \frac{\phi_n}{\sqrt{2\pi R}}$

Unification of forces from 5d (KK idea)

1) Gauge field $\Rightarrow$ gauge + scalar

$$M = (\mu, 4) \longleftrightarrow (x^\mu, y) \qquad y \equiv x^4$$

$$A_M = \begin{pmatrix} A_\mu(x,y) \\ \phi(x,y) \end{pmatrix} \qquad \phi \equiv A_4$$

y-dependence : expand in series as before

$$F_{MN}^2 = F_{\mu\nu}^2 + 2F_{\mu 4}^2 \qquad F_{\mu 4} = \partial_\mu A_4 - \partial_4 A_\mu = \partial_\mu \phi - \partial_y A_\mu$$

5d gauge invariance : $\delta A_M = \partial_M \Lambda(x,y)$

$$\Rightarrow \delta A_\mu = \partial_\mu \Lambda \quad , \quad \delta\phi = \partial_y \Lambda$$

$$\Rightarrow \delta A_\mu^n = \partial_\mu \Lambda_n \quad , \quad \delta\phi_n = i\frac{n}{R}\Lambda_n$$

- $n=0 \Rightarrow$ usual 4d gauge invariance
- $n\neq 0 \Rightarrow$ gauge choice : $\phi_n = 0$ (A_μ^n massive)

longitudinal polarization

$$\Rightarrow F_{\mu 4}^n = \begin{cases} \frac{in}{R} A_\mu^n & n\neq 0 \\ \partial_\mu \phi_0 & n=0 \end{cases}$$

$$\Rightarrow \int dy \, \frac{1}{4} F_{MN}^2 \underset{2\pi R}{=} \frac{1}{4} F_{\mu\nu}^{(n)2} + \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2}\frac{n^2}{R^2} A_\mu^{(n)2}$$

$\Rightarrow$ 4d : $U(1)$ gauge field + massless scalar + massive KK-vectors $n = \pm 1, \pm 2, \ldots$ with masses^2 $(n/R)^2$

2) gravity 5d → gravity 4d + U(1) + scalar (radion)

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu 4} \\ G_{\mu 4} & G_{44} \end{pmatrix}(x,y)$$

$$\delta G_{MN} = G_{MP}\partial_N \xi^P + G_{NP}\partial_M \xi^P + \xi^P \partial_P G_{MN} \qquad \xi^M = \begin{pmatrix} \xi^\mu \\ \xi^4 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{array}{l} \delta G_{44} = 2 G_{4\mu}\partial_4 \xi^\mu + 2 G_{44}\partial_4 \xi^4 + \xi^\mu \partial_\mu G_{44} + \xi^4 \partial_4 G_{44} \\ \delta G_{\mu 4} = G_{\mu\nu}\partial_4 \xi^\nu + G_{\mu 4}\partial_4 \xi^4 + G_{4\nu}\partial_\mu \xi^\nu + G_{44}\partial_\mu \xi^4 \\ \qquad + \xi^\nu \partial_\nu G_{\mu 4} + \xi^4 \partial_4 G_{\mu 4} \end{array} \right.$$

δG_{44} : $\xi^\mu = 0 \Rightarrow \delta G_{44} = 2 G_{44}\partial_4 \xi^4 + \xi^4 \partial_4 G_{44}$

$$\delta G_{44}^n = \sum_m \left\{ 2 G_{44}^{n-m} \frac{im}{R} \xi_m^4 + \frac{i(n-m)}{R} \xi_m^4 G_{44}^{n-m} \right\}$$

$$= \sum_m \frac{i(n+m)}{R} \xi_m^4 G_{44}^{n-m}$$

$G_{44} = e^{2\phi} \Rightarrow \delta\phi = \partial_y \xi^4 + \xi^4 \partial_y \phi$

$$\Rightarrow \delta\phi_n = \frac{in}{R}\xi_n^4 + \sum_m \frac{im}{R}\phi_m \xi_{n-m}^4$$

non-trivial ξ_n^4, $n \neq 0$ gauge choice : $\phi_n = 0$ (or $G_{44}^n = 0$)

$\delta G_{\mu 4}$: $\xi^4 = 0 \Rightarrow \delta G_{\mu 4} = G_{\mu\nu}\partial_4 \xi^\nu + G_{4\nu}\partial_\mu \xi^\nu + \xi^\nu \partial_\nu G_{\mu 4}$

non-trivial ξ_n^μ, $n \neq 0$ gauge choice : $G_{\mu 4}^n = 0$

$\Rightarrow G_{\mu\nu}^n$: massive spin-2 (5 helicities : ± 2, $\underset{\uparrow\, G_{\mu 4}^n}{\pm 1}$, $\underset{\varphi\, G_{44}^n}{0}$)

with gauge choice $\Rightarrow$

$$G_{MN}(x,y) = e^{2\phi(x)} \begin{pmatrix} \hat{G}_{\mu\nu}(x,y) & A_\mu(x) \\ A_\nu(x) & 1 \end{pmatrix} \qquad \begin{array}{l} G_{44} = e^{2\phi} \\ G_{\mu 4} = e^{2\phi} A_\mu \end{array}$$

leftover gauge freedom $\xi^\mu(x)$, $\xi^4(x)$ $\qquad \delta x^\mu = -\xi^\mu$, $\delta y = -\xi^4$

$$\delta \hat{G}_{\mu\nu} = \delta\left(e^{-2\phi} G_{\mu\nu}\right) =$$

$$\delta G_{44} = \xi^\mu \partial_\mu G_{44} \Rightarrow \boxed{\delta\phi = \xi^\mu \partial_\mu \phi}$$

$$\delta G_{\mu 4} = G_{4\nu}\partial_\mu \xi^\nu + G_{44}\partial_\mu \xi^4 + \xi^\nu \partial_\nu G_{\mu 4} \Rightarrow$$

$$\delta A_\mu = \delta\left(e^{-2\phi} G_{\mu 4}\right) = e^{-2\phi}\delta G_{\mu 4} - 2 e^{-2\phi} G_{\mu 4}\delta\phi$$

$$= A_\nu \partial_\mu \xi^\nu + \partial_\mu \xi^4 + \underbrace{e^{-2\phi}\xi^\nu \partial_\nu\left(e^{2\phi}A_\mu\right)}_{\xi^\nu\partial_\nu A_\mu + \cancel{2A_\mu \xi^\nu \partial_\nu \phi}} - \cancel{2A_\mu \xi^\nu \partial_\nu \phi}$$

$$\Rightarrow \boxed{\delta A_\mu = A_\nu \partial_\mu \xi^\nu + \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^4}$$

$$\delta G_{\mu\nu} = G_{\mu\rho}\partial_\nu \xi^\rho + G_{\nu\rho}\partial_\mu \xi^\rho + \xi^\rho \partial_\rho G_{\mu\nu}$$

$$= G_{(\mu\lambda}\partial_{\nu)}\xi^\lambda + \xi^\lambda \partial_\lambda G_{\mu\nu} + G_{(\mu 4}\partial_{\nu)}\xi^4 + \xi^4 \partial_4 G_{\mu\nu}$$

$$\Rightarrow \delta \hat{G}_{\mu\nu} = e^{-2\phi}\delta G_{\mu\nu} - 2e^{-2\phi} G_{\mu\nu}\delta\phi$$

$$= \hat{G}_{(\mu\lambda}\partial_{\nu)}\xi^\lambda + \xi^\lambda \partial_\lambda \hat{G}_{\mu\nu} + \cancel{2(\xi^\lambda\partial_\lambda\phi)\hat{G}_{\mu\nu}} - \cancel{2\hat{G}_{\mu\nu}\xi^\lambda\partial_\lambda\phi}$$

$$+ A_{(\mu}\partial_{\nu)}\xi^4 + \xi^4 \partial_4 \hat{G}_{\mu\nu}$$

$$\Rightarrow \boxed{\delta \hat{G}_{\mu\nu} = \hat{G}_{(\mu\lambda}\partial_{\nu)}\xi^\lambda + \xi^\lambda\partial_\lambda \hat{G}_{\mu\nu} + A_{(\mu}\partial_{\nu)}\xi^4 + \xi^4\partial_4\hat{G}_{\mu\nu}} \qquad [\,(\mu,\nu) = \mu\nu + \nu\mu\,]$$

$\Rightarrow$ $\xi^\mu(x)$: 4d diffeomorphisms (from 5d diffeo's), $\xi^4(x)$: 4d $U(1)$ gauge transformations

$$ds^2 = e^{2\phi}\, d\hat{s}^2$$

$$d\hat{s}^2 = \hat{G}_{\mu\nu}\, dx^\mu dx^\nu + 2A_\mu dx^\mu dy + dy\, dy$$

$$= \left(\hat{G}_{\mu\nu} - A_\mu A_\nu\right) dx^\mu dx^\nu + \left(dy + A_\mu dx^\mu\right)^2$$

$$\delta_4\left(dy + A_\mu dx^\mu\right) = -dx^\mu\, \partial_\mu \xi^4 + \partial_\mu \xi^4\, dx^\mu = 0$$

$$\delta_4\Big(\underbrace{\hat{G}_{\mu\nu} - A_\mu A_\nu}_{\hat{g}_{\mu\nu}}\Big) = \cancel{A_{(\mu}\partial_{\nu)}\xi^4} + \xi^4 \partial_4 \hat{G}_{\mu\nu} - \cancel{A_{(\mu}\partial_{\nu)}\xi^4} = \xi^4 \partial_4 \hat{G}_{\mu\nu} = 0$$

for $\hat{G}_{\mu\nu}$

(x, y)

$$\Rightarrow \quad G_{MN} = e^{2\phi} \begin{pmatrix} \hat{g}_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}$$

$$* \quad R\left[e^{2\phi} g_{\mu\nu}\right] = e^{-2\phi}\left\{R - 2(D-1) D^2\phi - (D-2)(D-1)(D\phi)^2\right\}$$

$$\sqrt{g}\left[e^{2\phi} g_{\mu\nu}\right] = e^{D\phi}\sqrt{g}$$

$$\Rightarrow \quad \sqrt{g}\, R^{(5)}\left[e^{2\phi}\hat{G}_{MN}\right] = e^{3\phi}\left\{\hat{R}^{(5)} - 8\hat{D}^2_\mu\phi - 12(\hat{D}_\mu\phi)^2\right\}$$

$$\left[\begin{array}{ll} R^\lambda{}_{\mu\nu\rho} = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\rho\sigma} - \Gamma^\sigma_{\mu\rho}\Gamma^\lambda_{\nu\sigma} & R_{\mu\rho} = R^\lambda{}_{\mu\lambda\rho} \\ \Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\sigma\nu}\left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right) & R = g^{\mu\rho} R^\lambda{}_{\mu\lambda\rho} \end{array}\right]$$

$$\bullet \quad \hat{G} = \begin{pmatrix} \hat{g} + A \otimes A^T & A \\ A^T & 1 \end{pmatrix} = \begin{pmatrix} \vec{g}_1 + \vec{A} A_1 & \vec{g}_2 + \vec{A} A_2 & \cdots & \vec{g}_n + \vec{A} A_n & \vec{A} \\ A_1 & A_2 & & A_n & 1 \end{pmatrix}$$

$$\vec{g}_i \equiv \begin{pmatrix} \hat{g}_{1i} \\ \hat{g}_{2i} \\ \vdots \\ \hat{g}_{ni} \end{pmatrix}$$

1) $$\det \begin{pmatrix} \vec g_1 + \vec A A_1 & \vec g_2 + \vec A A_2 & \cdots & \vec g_n + \vec A A_n & \vec A \\ A_1 & A_2 & & A_n & 1 \end{pmatrix} = \det \hat G$$

$$= \det \begin{pmatrix} \vec g_1 & \cdots & \vec g_n & \vec A \\ 0 & & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} \hat g & A \\ & 1 \end{pmatrix} = \det \hat g$$

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2) $$\det\left(\hat g + A \otimes A^T\right) = \det \hat g + \det\left(1 + \hat g^{-1} A \otimes A^T\right)$$

$$\det\left(1 + B \otimes A^T\right) = \det\left(\vec e_1 + A_1 \vec B \,,\ \vec e_2 + A_2 \vec B \,,\ \cdots\ \vec e_n + A_n \vec B\right)$$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{i-th line}$$

$$= \det\left(\vec e_1 + A_1 \vec B \,,\ \vec e_2 - \frac{A_2}{A_1} \vec e_1 \,,\ \cdots ,\ \vec e_n - \frac{A_n}{A_1} \vec e_1\right)$$

$$= \underbrace{\begin{vmatrix} 1 + A_1 B_1 & -A_2/A_1 & \cdots & -A_n/A_1 \\ A_1 B_2 & 1 & & 0 \\ \vdots & & & \vdots \\ A_1 B_n & 0 & & 1 \end{vmatrix}}_{M_n} = -(-)^n \frac{A_n}{A_1} \begin{vmatrix} A_1 B_2 & 1 & & \\ \vdots & & & \\ & & & 1 \\ A_1 B_n & & & 0 \end{vmatrix}$$

$$+ \underbrace{\begin{vmatrix} 1 + A_1 B_1 & \cdots & -\frac{A_{n-1}}{A_1} \\ \vdots & & \\ A_1 B_{n-1} & & 1 \end{vmatrix}}_{M_{n-1}}$$

$$\Rightarrow \det M_n = \det M_{n-1} - (-)^n \frac{A_n}{A_1} (-)^{n-1} A_1 B_n \det 1 = \det M_{n-1} + A_n B_n$$

$$= \cdots = \vec A \cdot \vec B$$

$$\Rightarrow \det\left(1+\hat g^{-1} A\otimes A^T\right) = A^T \hat g^{-1} A$$

$$\Rightarrow \det\left(\hat g + A\otimes A^T\right) = \det g + A^T \hat g^{-1} A$$

$$\Rightarrow \hat G^{-1} = \begin{pmatrix} \hat g^{-1} & -\hat g^{-1}A \\ -A^T \hat g^{-1} & 1+A^T\hat g^{-1}A \end{pmatrix}$$

$$* \quad \begin{pmatrix} \hat g + A\otimes A^T & A \\ A^T & 1 \end{pmatrix} \begin{pmatrix} \hat g^{-1} & -\hat g^{-1}A \\ -A^T \hat g^{-1} & 1+A^T\hat g^{-1}A \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + \cancel{A\otimes A^T\hat g^{-1}} - \cancel{A\otimes A^T\hat g^{-1}} & \cancel{-A} - A\cancel{A^T\hat g^{-1}A} + \cancel{A} + A\cancel{A^T\hat g^{-1}A} \\ \cancel{A^T\hat g^{-1}} - \cancel{A^T\hat g^{-1}} & -\cancel{A^T\hat g^{-1}A} + 1 + \cancel{A^T\hat g^{-1}A} \end{pmatrix} \quad \text{ok}$$

fields: g-independent

$$\Gamma^\sigma_{\mu\nu} = \hat\Gamma^\sigma_{\mu\nu} + \frac12 \hat G^{\sigma\nu}\left\{\partial_\mu(A_\lambda A_\nu) + \partial_\lambda(A_\mu A_\nu) - \partial_\nu(A_\mu A_\lambda)\right\} + \frac12 \hat G^{\sigma 4}\left(\partial_\mu A_\lambda + \partial_\lambda A_\mu - \cancel{\partial_4 \hat g_{\mu\lambda}}\right)$$

$$= \hat\Gamma^\sigma_{\mu\nu} - \frac12 A^\sigma \cancel{\left(\partial_\mu A_\lambda + \partial_\lambda A_\mu\right)} + \frac12 \hat g^{\sigma\nu}\underbrace{\left(A_\nu\partial_\mu A_\lambda + A_\lambda\partial_\mu A_\nu + A_\mu\partial_\lambda A_\nu + A_\nu\partial_\lambda A_\mu - A_\mu\partial_\nu A_\lambda - A_\lambda\partial_\nu A_\mu\right)}_{A_\lambda F_{\mu\nu} + A_\mu F_{\lambda\nu} + \cancel{A_\nu(\partial_\mu A_\lambda + \partial_\lambda A_\mu)}}$$

$$\Rightarrow \Gamma^\sigma_{\mu\lambda} = \hat\Gamma^\sigma_{\mu\lambda} + \frac12 \hat g^{\sigma\nu}\left(A_\lambda F_{\mu\nu} + A_\mu F_{\lambda\nu}\right)$$

$$\Gamma^\Sigma_{44} = \frac12 \hat G^{\Sigma N}\left(2\partial_4 \hat G_{4N} - \partial_N \hat G_{44}\right) = 0$$

$$\Gamma^\Sigma_{4\mu} = \frac12 \hat G^{\Sigma N}\left(\cancel{\partial_4 \hat G_{\mu N}} + \partial_\mu \hat G_{4N} - \partial_N \hat G_{\mu 4}\right) = \frac12 \hat G^{\Sigma\nu} F_{\mu\nu} \Rightarrow \begin{cases} \Gamma^\sigma_{4\mu} = \frac12 \hat g^{\sigma\nu} F_{\mu\nu} \\ \Gamma^4_{4\mu} = -\frac12 A^\nu F_{\mu\nu} \end{cases}$$

$$\Gamma^4_{\mu\lambda} = \frac12 \hat G^{44}\left(\partial_\lambda A_\mu + \partial_\mu A_\lambda\right) + \frac12 \hat G^{4\nu}\left(\partial_\lambda \hat G_{\mu\nu} + \partial_\mu \hat G_{\lambda\nu} - \partial_\nu \hat G_{\mu\lambda}\right)$$

$$= \frac12\left(1+\cancel{A^2}\right)\left(\partial_\mu A_\lambda + \partial_\lambda A_\mu\right) - \frac12 A^\nu \hat g_{\rho\nu}\hat\Gamma^\rho_{\mu\lambda} - \frac12 A^\nu \underbrace{\left\{\partial_\lambda(A_\mu A_\nu) + \partial_\mu(A_\lambda A_\nu) - \partial_\nu(A_\mu A_\lambda)\right\}}_{\cancel{A_\nu(\partial_\lambda A_\mu + \partial_\mu A_\lambda)} + A_\mu F_{\lambda\nu} + A_\lambda F_{\mu\nu}}$$

$$= -\frac12 A_\rho \hat\Gamma^\rho_{\mu\lambda} + \frac12\left(\partial_\mu A_\lambda + \partial_\lambda A_\mu\right) - \frac12 A^\nu\left(A_\lambda F_{\mu\nu} + A_\mu F_{\lambda\nu}\right)$$

$$\Gamma^N_{N4} = \Gamma^\mu_{\mu 4} + \cancel{\Gamma^4_{44}} = \frac{1}{2}\,\delta^\mu_\sigma\,\hat g^{\sigma\nu} F_{\mu\nu} = 0$$

$$\Gamma^N_{N\sigma} = \Gamma^\mu_{\mu\sigma} + \Gamma^4_{4\sigma} = \hat\Gamma^\nu_{\nu\sigma} + \cancel{\frac{1}{2} A^\nu F_{\sigma\nu}} - \cancel{\frac{1}{2} A^\nu F_{\sigma\nu}} = \hat\Gamma^\nu_{\nu\sigma}$$

$$R = \hat G^{M\rho}\,\partial_\rho \Gamma^N_{MN} - \hat G^{MR}\,\partial_\nu \Gamma^\nu_{MR} + \hat G^{MR}\left(\Gamma^\Sigma_{MN}\Gamma^N_{R\Sigma} - \Gamma^\Sigma_{MR}\Gamma^N_{N\Sigma}\right)$$

$$= \hat g^{\mu\rho}\,\partial_\rho \hat\Gamma^\nu_{\mu\nu} - \hat g^{\mu\rho}\,\partial_\nu \Gamma^\nu_{\mu\rho} - 2\hat G^{\mu 4}\,\partial_\nu \Gamma^\nu_{\mu 4} - \hat G^{MR}\,\Gamma^\sigma_{MR}\,\Gamma^N_{N\sigma} + \underbrace{\hat G^{MR}\,\Gamma^\Sigma_{MN}\,\Gamma^N_{R\Sigma}}$$

$$= \hat R - \hat g^{\mu\lambda}\,\partial_\sigma\left\{\hat g^{\sigma\nu} A_\lambda F_{\mu\nu}\right\} + A^\mu\,\partial_\sigma\left(\hat g^{\sigma\nu} F_{\mu\nu}\right) - 2\hat G^{\mu 4}\,\Gamma^\sigma_{\mu 4}\,\hat\Gamma^\nu_{\nu\sigma} - \hat g^{\mu\rho}\,\delta\Gamma^\sigma_{\mu\rho}\,\hat\Gamma^\nu_{\nu\sigma} + \dots$$

$$= \hat R - \hat g^{\mu\lambda}\,\hat g^{\sigma\nu}\,(\partial_\sigma A_\lambda) F_{\mu\nu} + \cancel{A^\mu\,\hat g^{\sigma\lambda} F_{\mu\lambda}\,\hat\Gamma^\nu_{\nu\sigma}} - \cancel{\hat g^{\mu\lambda}\,\hat g^{\sigma\rho} A_\lambda F_{\mu\rho}\,\hat\Gamma^\nu_{\nu\sigma}} + \dots$$

$$= \hat R + \frac{1}{2} F^2 + \underbrace{\hat G^{MR}\left\{\Gamma^\sigma_{MN}\Gamma^N_{R\sigma} + \Gamma^4_{MN}\Gamma^N_{R4}\right\}}_{\Gamma^\sigma_{M\nu}\Gamma^\nu_{R\sigma} + 2\Gamma^\sigma_{M4}\Gamma^4_{R\sigma} + \Gamma^4_{M4}\Gamma^4_{R4}}$$

$$\hat\varepsilon\{\ \} = \hat G^{44}\,\Gamma^\sigma_{4\nu}\Gamma^\nu_{4\sigma} + \hat G^{\mu 4}\left(2\Gamma^\sigma_{\mu\nu}\Gamma^\nu_{4\sigma} + 2\Gamma^\sigma_{\mu 4}\Gamma^4_{4\sigma}\right) + \left(\Gamma^\sigma_{\mu\nu}\Gamma^\nu_{\rho\sigma} + 2\Gamma^\sigma_{\mu 4}\Gamma^4_{\rho\sigma} + \Gamma^4_{\mu 4}\Gamma^4_{\rho 4}\right)\hat g^{\mu\rho}$$

$$= (1+A^2)\frac{1}{4}\,\hat g^{\sigma\lambda} F_{\nu\lambda}\,\hat g^{\nu\rho} F_{\sigma\rho} - A^\mu\left(\hat g^{\nu\lambda} F_{\sigma\lambda}\,\Gamma^\sigma_{\mu\nu} - \frac{1}{2}\,\hat g^{\sigma\lambda} \underline{F_{\mu\lambda}}\, A^\rho F_{\sigma\rho}\right) + \Gamma^\sigma_{\mu\nu}\Gamma^\nu_{\rho\sigma}\,\hat g^{\mu\rho}$$

$$+ \cancel{\hat g^{\mu\rho}\,\hat g^{\sigma\nu} F_{\mu\nu}\,\Gamma^4_{\rho\sigma}} + \frac{1}{4} A^\nu \underline{F_{\mu\nu}}\, A^\lambda F_{\rho\lambda}\,\hat g^{\mu\rho}$$

$$= -(1+A^2)\frac{1}{4} F^2 - \frac{1}{4}\left(A^\nu F_{\mu\nu}\right)^2 - A^\mu\,\cancel{\hat\Gamma^\sigma_{\mu\nu}\,\hat g^{\nu\lambda} F_{\sigma\lambda}} - \frac{1}{2}\,\hat g^{\nu\lambda} F_{\sigma\lambda}\, A^\mu\,\hat g^{\sigma\rho}\left(A_\mu F_{\nu\rho} + A_\nu F_{\mu\rho}\right)$$

$$+ \hat g^{\mu\rho}\,\hat\Gamma^\nu_{\rho\sigma}\,\hat g^{\sigma\lambda}\cancel{\left(A_\mu F_{\nu\lambda} + A_\nu F_{\mu\lambda}\right)} + \frac{1}{4}\,\hat g^{\mu\rho}\,\hat g^{\sigma\lambda}\left(A_\mu F_{\nu\lambda} + A_\nu F_{\mu\lambda}\right)\hat g^{\nu\tau} \times \left(A_\rho F_{\sigma\tau} + A_\sigma F_{\rho\tau}\right)$$

$$= -\frac{1}{4}\left(1+\cancel{A^2}\right)F^2 - \cancel{\frac{1}{4}\left(A^\nu F_{\mu\nu}\right)^2} + \cancel{\frac{1}{2}A^2}F^2 + \cancel{\frac{1}{2}\left(A^\nu F_{\mu\nu}\right)^2} - \cancel{\frac{1}{4}A^2} F^2 - \cancel{\frac{1}{2}\left(A^\nu F_{\mu\nu}\right)^2} + \cancel{\frac{1}{4}\left(A^\nu F_{\mu\nu}\right)^2}$$

$$= -\frac{1}{4} F^2$$

$$\Rightarrow \hat R^{(5)} = \hat R^{(4)} + \frac{1}{4} F^2$$

$$\sqrt{g}\, R^{(5)}\left[e^{2\phi}\hat{G}_{\mu\nu}\right] \underset{4d}{=} e^{3\phi}\sqrt{\hat{G}}\left[\hat{R}^{(4)} + \frac{1}{4}F^2 - 8(\hat{D}\phi)^2 - 12(\hat{D}\phi)^2\right] \qquad (*)$$

$$\Downarrow \quad 24(\hat{D}\phi)^2$$

$$\underset{4d}{=} e^{3\phi}\sqrt{\hat{G}}\left[\hat{R} + \frac{1}{4}F^2 + 12(\hat{D}\phi)^2\right]$$

$$= 12 e^{2\omega}(D\phi)^2$$

$$\hat{G}_{\mu\nu} = e^{2\omega} g_{\mu\nu} \Rightarrow \qquad \| \qquad e^{2\omega}\left[R - 6 D^2\omega - 6(D\omega)^2\right]$$

$$\omega = -\frac{3}{2}\phi \Rightarrow \qquad = R - 6\cdot\frac{9}{4}(D\phi)^2 + 12(D\phi)^2 + \frac{1}{4}e^{3\phi}F^2$$

$$12 - \frac{27}{2} = -\frac{3}{2} \qquad = R - \frac{3}{2}(\partial\phi)^2 + \frac{1}{4}e^{3\phi}F^2$$

$$\phi = \frac{1}{\sqrt{3}}\varphi \qquad = R - \frac{1}{2}(\partial\varphi)^2 + \frac{1}{4}e^{\sqrt{3}\varphi}F^2$$

$$G_{MN} = e^{2/\sqrt{3}\,\varphi}\begin{pmatrix} e^{-\sqrt{3}\varphi} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2\kappa_5^2}\int dy\, R^{(5)} = \frac{2\pi R}{2\kappa_5^2}\left\{ R^{(4)} - \frac{1}{2}(\partial\varphi)^2 + \frac{1}{4}e^{\sqrt{3}\varphi}F^2 + \sum_{n\neq 0}^{KK}(\text{massive spin-2 fields})\right\}$$

Kaluza-Klein idea: unify gravity with electromagnetism in 5d

φ: "Radion" excitation

Radion: • $G_{44} = e^{2\phi}$ $\quad (*) \Rightarrow \langle e^{3\phi}\rangle$ changes $2\pi R$

$\Rightarrow \langle e^{\sqrt{3}\varphi}\rangle$ fixes the radius of the extra dimension

EFT: integrate out massive KK modes $\Rightarrow$ EFT for light modes

$\Rightarrow$ higher dim eff operators

4d $\leftrightarrow$ 5d couplings $\quad$ 1) U(1): $\quad \frac{1}{g_4^2} = \frac{2\pi R}{g_5^2} \qquad [g_5^2] =$ length

2) gravity: $\quad M_p^2 = \frac{1}{\kappa_4^2} = \frac{2\pi R}{\kappa_5^2} = 2\pi R M_5^3$

- Forces in extra dims

1) Coulomb potential

$\vec{B}=0 \quad \rho = Q\,\delta^3(x) \qquad \vec{\nabla}\vec{E}=\rho \quad \vec{\nabla}\times\vec{E}=0 \quad \Rightarrow \quad \vec{E}=\vec{\nabla}\phi\,, \quad \vec{\nabla}^2\phi=\rho$

$\phi \sim Q \int d^3k \, \frac{e^{i\vec{k}\vec{x}}}{k^2} \sim \frac{1}{r} \qquad r=|\vec{x}| \qquad$ (by dim analysis)

2) Newtonian potential : same with $Q \to GM^2$

n extra dim : $\int d^3k \to \int d^{3+n}k \quad \Rightarrow \quad \frac{1}{r} \to \frac{1}{r^{1+n}}$

3d → (3+n)d due to KK excitations

massive particle:

$$\int d^3k \, \frac{e^{i\vec{k}\vec{x}}}{k^2+m^2} \approx \int dk\, k^2 \int_{-1}^{1} d\cos\theta \, \frac{e^{ikr\cos\theta}}{k^2+m^2} \sim \frac{1}{r} \underbrace{\int_0^\infty dk \, \frac{k \sin kr}{k^2+m^2}}_{\sim\, e^{-mr}} \qquad m>0$$

$\Rightarrow$ Yukawa potential : $\frac{e^{-mr}}{r} \qquad m = 0, \underbrace{\frac{1}{R}, \frac{2}{R}, \ldots}_{\text{multiplicity } 2}$

$$\Rightarrow \frac{1}{r}\left\{1+2e^{-r/R}+2e^{-2r/R}+\ldots\right\} \equiv V$$

$R\to 0 \Rightarrow V \to \frac{1}{r}$ (3d)

R finite $\Rightarrow V \approx \frac{1}{r}\left\{1+2\left(\frac{1}{1-e^{-r/R}}-1\right)\right\} = \frac{1}{r}\left\{1+2\,\frac{e^{-r/R}}{1-e^{-r/R}}\right\}$

$R\to\infty \Rightarrow V \sim \frac{2R}{r^2} \qquad Q \to Q_5 = 2R\,Q_4$