Graduate course on flavor physics

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Abstract

The study of B-meson decays is an active research topics in elementary particle physics, both in experiment and theory. By confronting experimental and theoretical results, several parameters of the standard model (SM)lagrangian can be determined, in particular the elements of the Cabibbo-Kobajashi-Maskawa (CKM) matrix.

An important class of B-meson decays are the so-called rare dacay, which by definition are induced in the SM only at the one-loop level. As such, they are particularly sensitive to physics beyond the SM.

The first four lectures will be given by Christoph Greub. We will first briefly review the part of the SM which is relevant for flavor physics. Then we will look at inclusive tree-level decays (including QCD corrections). Then we will discuss the effective theory formulation of B decays (effective Hamiltonian formalism, matching, renormalization group equations and all that). We will apply these techniques for rare inclusive B-decays.

Javier Virto will then give four lectures on exclusive B-meson decays and their application for the extraction of SM parameters and searches for new physics. We will start with a classification of the different processes, and explain the techniques necessary for the calculation of the amplitudes in each case, having in mind that we are dealing with hadronic processes, where non-perturbative effects are important. These amplitudes will then be used in a few phenomenological applications, including the understanding of the current anomalies in $b \to s$ exclusive transitions.

Literature

General

- O. Nachtmann, *Elementarteilchenphysik, Phänomene und Konzepte (German and English editions)*
- M. Peskin, D. Schröder, *An introduction to Quantum field theory*
- T. Cheng, L. Li, *Gauge theory of elementary particle physics*
- J. Donoghue, E. Golowich, B. Holstein, *Dynamics of the Standard Model*

More specific

- G. Buchalla, A. Buras, M. Lautenbacher, *Weak decays beyond leading logarithms*, hep-ph/9512380. Review article.
- A. Buras, *Weak Hamiltonians, CP Violation and Rare decays*, hep-ph/9806471. Excellent pedagogical review article.
- C. Jarlskog, CP Violation: Book with articles from several authors.

Remarks

The first part is a short version of a previous course by C. Greub. We thank Jason Aebischer for converting hand-written notes into a version in LaTex.

Contents

1 Introduction

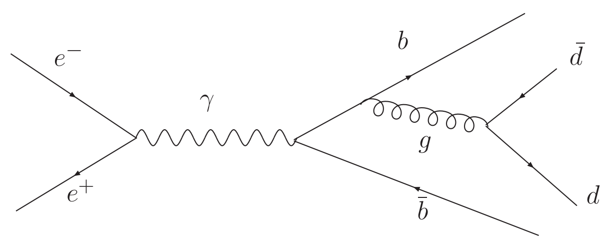
We briefly look at the experiments in which b-quarks are produced; we then look at the main decay mechanisms of b-quarks.

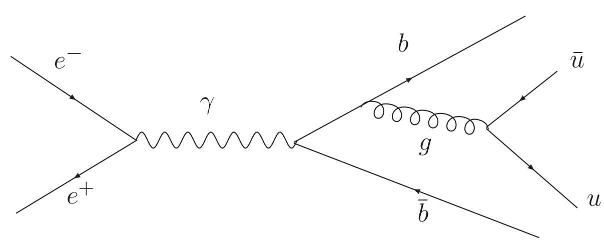
1.1 B Physics experiments

1.1.1 B-Mesons produced in e^+e^- collisions

$$e^+e^- \to \Upsilon(4s) \longrightarrow \left\{ \begin{array}{l} B_d^0\,\bar{B}_d^0 \sim 50\% \\ B^+B^- \sim 50\% \end{array} \right\} \qquad \Upsilon \equiv \text{called Upsilon}$$

$\Upsilon(4s)$: $(b\bar{b})$ resonance





$\sqrt{s} = m(\Upsilon(4s)) = (10.5800 \pm 0.0035)$ GeV

$m_{B^0} = (5.2792 \pm 0.0018)$ GeV

$m_{B^\pm} = (5.2789 \pm 0.0018)$ GeV

$m_{B_s} = (5.3693 \pm 0.002)$ GeV, $\quad B_s = (\bar{b}s)$

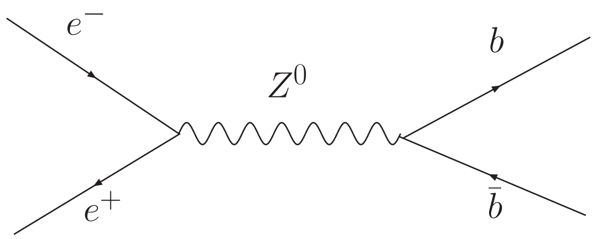
Therefore: $\Upsilon(4s) \to B_s\bar{B}_s$ kinematically not possible

$\Rightarrow$ The experiments using $\Upsilon(4s)$ work with a pure sample of B-mesons!

$\Upsilon(4s)$ Experiments

ARGUS @ DESY	CLEO @ Cornell	BABAR @ SLAC	BELLE @ KEK	BELLE upgrade planed
stopped $\sim$ 1992	$\sim$1990-2000	$\sim$2000-2008	$\sim$2000-2010	start 2018 (?)
evidence that m_t is heavy (through $B\bar{B}$ mixing)	few$\times 10^6 B\bar{B}$ events; many interesting results	few$\times 10^9 B\bar{B}$ events; many results on CP-violation; rare decays	similar performance as BABAR	higher luminosity, factor 50-100 more events

Also LEP (in particular ALEPH) was involved in B-Physics. This was not an experiment at $\Upsilon(4s)$ resonance, but a high energy experiment working at the Z^0 pole.



b and $\bar{b}$ hadronize subsequently.

$\bar{b} \to B^+ \sim 40\%$

$\bar{b} \to B^0 \sim 40\%$

$\bar{b} \to B_s^0 \sim 10\%$

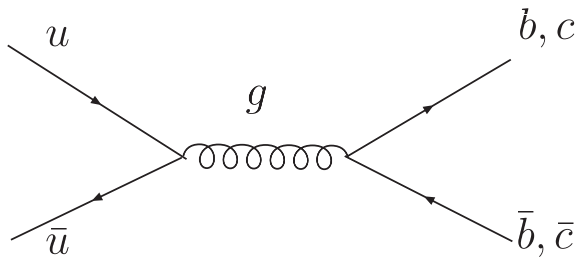
$\bar{b} \to \Lambda_b \sim 10\%$

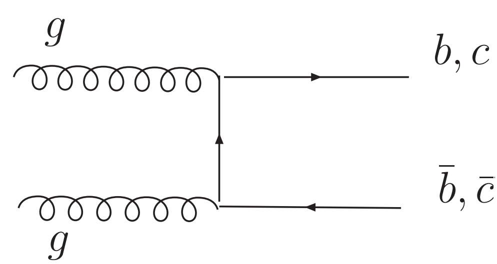
1.1.2 B-Mesons produced in hadron collisions

- LHCb = Large Hadron Collider beauty

  Specialized for studying decays of hadrons which contain a b-quark or a c-quark.

  Production of b-quarks (or c-quarks) via the strong interaction (production in pairs)





- TEVATRON @ Fermilab is a $p\bar{p}$ machine

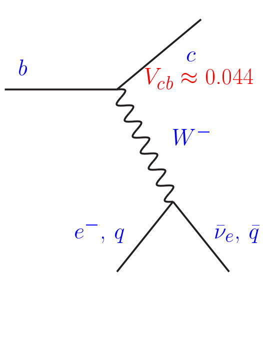
  $\rightarrow$ Production mechanism of b- and c-quarks is the same.

B-mesons (b-quarks) always decay through weak interactions.

We therefore will first repeat the electro-weak sector of the Standard Model (SM).

1.2 Decay modes of b-quarks; terminology

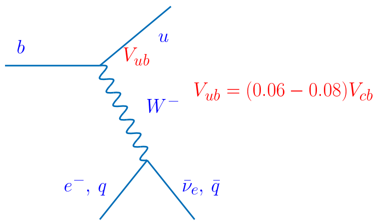
- Charged current decay (Cabibbo-allowed)



Inclusive: $B \rightarrow X_c \ell \nu_\ell$; Exclusive: $B \rightarrow D^{(*)} \ell \nu_\ell$ etc.

  This is a so-called tree-level decay. The b-quark decay induces the corresponding B-meson decays.

- Charged current decay (Cabibbo-suppressed)

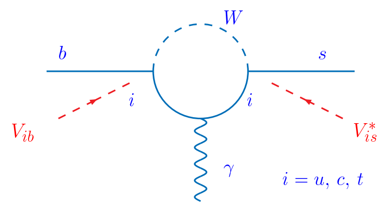


Inclusive: $B \to X_u \ell \nu_\ell$, Exclusive: $B \to (\pi, \rho)\ell\nu_\ell$ etc.

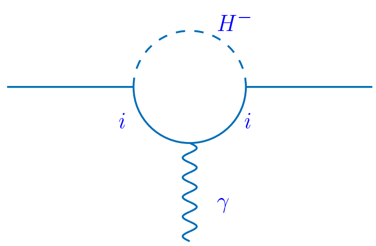
- Rare b-quark decays

  Rare B decays, like $b \to s\gamma$ or $b \to s\ell^+\ell^-$ are induced at the one-loop level in the SM. These processes test the SM at the QT level; they are sensitive to certain CKM matrix elements.

  A typical diagram is



  These processes are also very sensitive to extensions of the SM, (e.g. $H^\pm$, SUSY contrib., etc.):



<u>Terminology:</u>

Exclusive rare B-meson decays: $B \to K^*\gamma$, $B \to K^{(*)}\mu^+\mu^-$ etc., i.e. the final hadronic state (with strangeness) is a very specific one.

Inclusive rare B-meson decays: $B \to X_s\gamma$, $B \to X_s\mu^+\mu^-$ etc., i.e. X_s denote any hadronic final state with strangeness.

The theoretical predicition for inclusive decays is cleaner, because no description of the hadronzation of the produced s-quark is needed.

2 Standard Model briefly summarized

2.1 Gauge group

$G_{\mathrm{SM}} = SU(3)_c \times SU(2)_L \times U(1)_Y$

The group $SU(3)_c$ has 8 generators and therefore 8 real gauge bosons, the gluons G^A_μ $(A = 1, ..., 8)$.

The group $SU(2)_L$ has 3 generators and therefore 3 real gauge bosons, W^i_μ $(i = 1, 2, 3)$.

The group $U(1)_Y$ has 1 generator and therefore 1 real gauge boson, B_μ.

The "physical" gauge bosons $W^\pm_\mu$, Z^0_μ, A_μ are certain linear combinations of W^i_μ and B_μ.

2.2 Fundamental fermions

$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ u_R d_R	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$ c_R s_R Quarks	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$ t_R b_R

$\begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix}$ e_R	$\begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix}$ μ_R Leptons	$\begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix}$ τ_R

The notation indicates that the left-handed fields are doublets with respect to the group $SU(2)_L$, while the right-handed fermions are singlets under this group.

Taking into account the color degree of freedom, each of the three generations contains 15 Weyl-fermions (two component objects):

$15 = 4 \cdot 3 + 3$

2.3 Electro-weak Interaction

$G_{\mathrm{EW}} = SU(2)_L \times U(1)_Y$ electroweak gauge group

The electro-weak Lagrangian L can be decomposed as

$$L = L_G + L_{fG} + L_\phi + L_{\mathrm{Yuk}}\,.$$

We now go through the various pieces.

2.3.1 Part L_G

$$L_G = -\frac{1}{4} W^i_{\mu\nu} W^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$
$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\epsilon^{ijk} W^j_\mu W^k_\nu,\ i = 1,2,3$$
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

2.3.2 Part L_{fG}

$$L_{fG} = \bar{\Psi} i\gamma^\mu D_\mu \Psi$$

with

$$D_\mu \Psi = (\partial_\mu + igT^i W^i_\mu + ig'\frac{Y}{2} B_\mu)\Psi \qquad \text{covariant derivative}$$

Ψ means either a left-handed doublet or a right-handed singlet, as detailed in the following:

- Left-handed Doublet, e.g. $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$

  $$T^i = \frac{1}{2}\tau^i \qquad (\tau^i\text{: Pauli matrices})$$

  As the gauge group G_{EW} is a direct product, the $SU(2)$ generators T^i have to commute with the $U(1)$ generator Y:

  $$[T^i, Y] = 0\,.$$

  As a consequence, Y has the form

  $$Y = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

This means that u_L, d_L have the same hypercharge y.

Fix the hypercharges such that the electric charge Q is a linear combination of T^3 and Y:

$$Q = T^3 + \frac{Y}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} .$$

$$u : \frac{2}{3} = \frac{1}{2} + \frac{y}{2} \rightarrow y = \frac{1}{3}$$

$$d : -\frac{1}{3} = -\frac{1}{2} + \frac{y}{2} \rightarrow y = \frac{1}{3}$$

Analogously:

$$\begin{pmatrix} u \\ d \end{pmatrix}_{L,\frac{1}{3}} \quad \begin{pmatrix} c \\ s \end{pmatrix}_{L,\frac{1}{3}} \quad \begin{pmatrix} t \\ b \end{pmatrix}_{L,\frac{1}{3}}$$

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L,-1} \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_{L,-1} \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_{L,-1}$$

$$D_\mu \begin{pmatrix} u \\ d \end{pmatrix}_L = \left(\partial_\mu + ig\frac{\tau^i}{2}W^i_\mu + ig'\frac{1}{6}B_\mu\right)\begin{pmatrix} u \\ d \end{pmatrix}_L$$

- Right-handed Singlet, e.g. u_R

$$T^i u_R = 0 \qquad (\leftrightarrow u_R \text{ is a singlet w.r.t. } SU(2))$$

Therefore, the hypercharge assignments are

$$(u_R)_{\frac{4}{3}}, (c_R)_{\frac{4}{3}}, (t_R)_{\frac{4}{3}}$$

$$(d_R)_{-\frac{2}{3}}, (s_R)_{-\frac{2}{3}}, (b_R)_{-\frac{2}{3}}$$

$$(e_R)_{-2}, (\mu_R)_{-2}, (\tau_R)_{-2}$$

$$D_\mu u_R = \left(\partial_\mu + ig'\frac{2}{3}B_\mu\right) u_R \quad , \text{etc.}$$

2.3.3 Part L_ϕ

$$L_\phi = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi)$$

where $\phi = \begin{pmatrix} \phi^\dagger \\ \phi^0 \end{pmatrix}_1$ is the Higgs doublet (in the SM we only have one Higgs doublet). The subscript denotes the hypercharge quantum number.

$$D_\mu \phi = (\partial_\mu + ig\frac{\tau^i}{2} W_\mu^i + ig'\frac{1}{2} B_\mu)\phi$$

$$V(\phi) = -\mu^2(\phi^\dagger \phi) + \lambda(\phi^\dagger \phi)^2 \qquad \text{Higgs potential}$$

L_ϕ is invariant under $SU(2)_L \times U(1)_Y$.

The potential has a minimum <u>not</u> for $\phi = 0$, but e.g. for the constant field configuration

$$\phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} ; \qquad v = \sqrt{\frac{\mu^2}{\lambda}}$$

<u>Unitary gauge:</u> The Higgs doublet $\phi(x) = \begin{pmatrix} \phi^\dagger(x) \\ \phi^0(x) \end{pmatrix}$ can be written in the form

$$\phi(x) = U^{-1}(x) \begin{pmatrix} 0 \\ \frac{H(x)}{\sqrt{2}} \end{pmatrix} ,$$

where $H(x)$ is a real field and $U^{-1}(x)$ is a (x-dependent) $SU(2)$ matrix.

Now do the local $SU(2)$ gauge transformation associated with $U(x)$ everywhere in the Lagrangian: The form of the Lagrangian stays the same, except that ϕ has the form

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{H(x)}{\sqrt{2}} \end{pmatrix} . \tag{2.1}$$

We say, that we are in the <u>unitary gauge</u>, after doing this gauge transformation.

The configuration which realizes the minimum of the Higgs potential is then

$$\phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} .$$

In other words, $H(x)$ has a vaccum expectation value, v. We therefore can write $H(x) = v + \eta(x)$, where $\eta(x)$ has zero vacuum expectation value. $\eta(x)$ is the physical Higgs field.

To summarize: The Higgs doublet in the unitary gauge has the form

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} .$$

2.3.4 Part L_{Yuk}

$$L_{\text{Yuk}} = h^e_{ij}\, \bar{l}_L^{\,i}\, \phi\, e^j_R + h^d_{ij}\, \bar{q}^i_L\, \phi\, d^j_R + h^u_{ij}\, \bar{q}^i_L\, \tilde{\phi}\, u^j_R + h.c. \tag{2.2}$$

- q^i_L left-handed quark doublet of i^{th} generation
- l^i_L left-handed lepton doublet of i^{th} generation
- $\tilde{\phi} := i\tau^2\phi^*; h^e, h^d, h^u$: arbitrary 3×3 matrices called Yukawa couplings
- $\tilde{\phi}$ transforms in the same way under $SU(2)_L$ as ϕ
  → Exercise!

$\underline{SU(2)_L}$: Invariance of L_{Yuk} is then obvious

$\underline{U(1)_Y}$: Invariance of L_{Yuk} is again easy to see

$\bar{l}_L\,\phi\, e_R$	$\bar{q}_L\,\phi\, d_R$	$\bar{q}_L\,\tilde{\phi}\, u_R$
$1+1-2=0$	$-\frac{1}{3}+1-\frac{2}{3}=0$	$-\frac{1}{3}-1+\frac{4}{3}=0$

2.3.5 Masses of gauge bosons (W,Z-bosons)

$(D_\mu\phi)(D^\mu\phi)$ in L_ϕ contains terms of the form: $v \cdot v \cdot Boson \cdot Boson$. These bilinear terms (in the boson fields) are the Boson mass terms.

Result for boson masses:

$$L_{\text{Boson mass}} = m^2_W W^+_\mu W^{\mu-} + \frac{1}{2} m^2_Z Z_\mu Z^\mu + \underbrace{0 \cdot A_\mu A^\mu}_{\text{Photon massles}} \tag{2.3}$$

$$W^\pm_\mu = \left(W^1_\mu \mp i W^2_\mu\right)/\sqrt{2} \tag{2.4}$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{2.5}$$

When doing this calculation, one finds that

$$m_W^2 = \frac{g^2 v^2}{4}\,, \quad m_Z^2 = \frac{v^2(g^2+g'^2)}{4}\,, \quad \tan\theta_W = \frac{g'}{g}\,.$$

$$\Rightarrow m_Z^2 = \frac{v^2 g^2}{4}\left(1+\tan^2\theta_W\right) = \frac{v^2 g^2}{4\cos^2\theta_W} = \frac{m_W^2}{\cos^2\theta_W}$$

$$m_W = (80.41 \pm 0.10)\,\mathrm{GeV}\,; \qquad m_Z = (91.187 \pm 0.007)\,\mathrm{GeV}$$

2.3.6 L_{fG} expressed in terms of $W_\mu^\pm, Z_\mu^0, A_\mu$

Start with $L_{fG} = \bar\psi i\gamma^\mu D_\mu \psi$ and express the original gauge bosons W_μ^i and B_μ in terms of the mass eigenstate bosons $W_\mu^\pm$, Z_μ and A_μ. Then write the result in the form

$$L_{fG} = L_{CC} + L_{NC} + L_{kin}$$

CC: charged current interactions; NC: neutral current interactions

One gets

$$L_{CC} = -\frac{g}{\sqrt{2}}\left[\bar e_L \gamma_\mu \nu_L W^{\mu-} + \bar d_L \gamma_\mu u_L W^{\mu-} + h.c.\right] + \text{ copy for 2. and 3. generation.}$$

$$L_{NC} = -eJ_\mu^{em} A^\mu + \frac{g}{\cos\theta_W} J_\mu^0 Z^\mu\,; \qquad e = g\sin\theta_W$$

$$J_\mu^{em} = Q_e \bar e\gamma_\mu e + Q_u \bar u \gamma_\mu u + Q_d \bar d\gamma_\mu d + \text{copy for 2. and 3. generation.}$$

$$J_\mu^0 = -\sum_f \left(g_L^f \bar f_L \gamma_\mu f_L + g_R^f \bar f_R \gamma_\mu f_R\right) + \text{copy for 2. and 3. generation..}$$

f runs over ν_e, e, u and d. The couplings $g_{L,R}^f$ read $g_{L,R}^f = T^3(f_{L,R}) - Q(f)\sin^2\theta_W$, which concretly means

$$g_L^\nu = \frac{1}{2};\ g_R^\nu = 0$$

$$g_L^e = -\frac{1}{2} + \sin^2\theta_W;\ g_R^e = \sin^2\theta_W$$

$$g_L^u = \frac{1}{2} - \frac{2}{3}\sin^2\theta_W;\ g_R^u = -\frac{2}{3}\sin^2\theta_W$$

$$g_L^d = -\frac{1}{2} + \frac{1}{3}\sin^2\theta_W;\ g_R^d = \frac{1}{3}\sin^2\theta_W$$

2.3.7 Fermion-Mass terms and the CKM Matrix

Start with L_{Yuk}. Replace there $\phi \to \phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$, getting

$$L_{mass}^{fermion} = \bar{e}_L^i M_{ij}^e e_R^j + \bar{d}_L^i M_{ij}^d d_R^j + \bar{u}_L^i M_{ij}^u u_R^j + h.c.$$

with

$$M_{ij}^e = \frac{v}{\sqrt{2}} h_{ij}^e \qquad \text{etc..}$$

For the following it is convenient to introduce the matrix notations

$$E_L = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}, \quad E_R = \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}, \quad D_L = \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \qquad \text{etc..}$$

leading to

$$L_{mass}^{fermion} = \bar{E}_L M^e E_R + \bar{D}_L M^d D_R + \bar{U}_L M^u U_R + \text{h.c..}$$

The mass matrices M^e, M^d, M^u are arbitrary complex 3×3 matrices.

Theorem: Let M be an arbitrary 3×3 matrix. It can always be written in the form

$$M = V_1 D V_2^\dagger,$$

where $V_1, V_2 \in U(3)$ and D diagonal (with non-negative entries).

Therefore we can write

$$L_{mass}^{fermion} = \underbrace{\bar{E}_L V_L^E}_{\bar{E}'_L} D^e \underbrace{V_R^{E\dagger} E_R}_{E'_R} + \ldots$$

D^e is the diagonal mass matrix for the charged leptons. E_L, E_R and E'_L, E'_R are called the weak- and the mass eigenstates, respectively. We have the relations

$$E_L = V_L^E E'_L\,, \qquad E_R = V_R^E E'_R$$

$$U_L = V_L^U U'_L\,, \qquad U_R = V_R^U U'_R$$

$$D_L = V_L^D D'_L\,, \qquad D_R = V_R^D D'_R$$

Look at charged current Lagrangian L_{CC}:

$$L_{CC} = -\frac{g}{\sqrt{2}} \left[\bar{E}_L \gamma_\mu N_L W^{\mu -} + \bar{D}_L \gamma_\mu U_L W^{\mu -} + h.c. \right]$$

$$L_{CC} = -\frac{g}{\sqrt{2}}\left[\bar{E}'_L\gamma_\mu V_L^{E\dagger}N_L W^{\mu-} + \bar{D}'_L\gamma_\mu V_L^{D\dagger}V_L^U U'_L W^{\mu-} + h.c.\right]$$

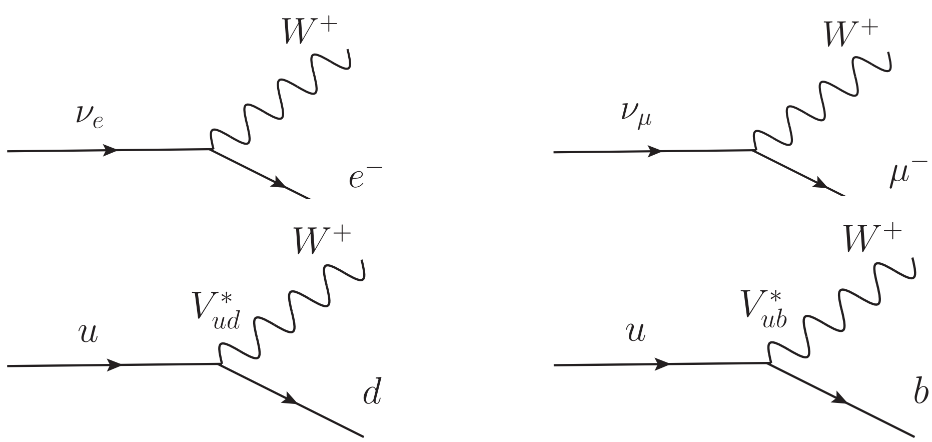
As we have no neutrino mass term, we can put $N_L = V_L^E N'_L$, leading to

$$L_{CC} = -\frac{g}{\sqrt{2}}\left[\bar{E}'_L\gamma_\mu N'_L W^{\mu-} + \bar{D}'_L\gamma_\mu \underbrace{V_L^{D\dagger}V_L^U}_{\doteq V^\dagger} U'_L W^{\mu-} + h.c.\right]$$

which implies that $V = V_L^{U\dagger}V_L^D$.

$V = V_L^{U\dagger}V_L^D$:	Cabibbo-Kobayashi-Maskawa matrix (CKM-Matrix)
3×3 unitary	

We get the following vertices:



We now turn to the neutral current interactions. It is easy to see that the matrices from change of basis drop out in L_{NC}.

This implies that there are $\underline{\underline{\text{no}}}$ $\underbrace{FCNC}_{\text{flavour changing neutral currents}}$ in the Standard Model at tree level!

For example, for the electromagnetic current of the charged leptons we get

$$\begin{aligned} J_\mu^{em} = &\quad Q_e\bar{E}_L\gamma_\mu E_L + Q_e\bar{E}_R\gamma_\mu E_R \\ = &\quad Q_e\bar{E}'_L\gamma_\mu V_L^{E\dagger}V_L^E E'_L + Q_e\bar{E}'_R\gamma_\mu V_R^{E\dagger}V_R^E E'_R \\ = &\quad Q_e\bar{E}'_L\gamma_\mu E'_L + Q_e\bar{E}'_R\gamma_\mu E'_R\,. \end{aligned} \tag{2.6}$$

2.3.8 Parameters in CKM Matrix V

$V \in U(N)\,; \quad N$: number of generations ($N = 3$ in reality)

- V has N^2 parameters (real way of counting)
- If V_{ij} was real (∀i,j), then $V \in O(N)$. $O(N)$ has $\frac{N(N-1)}{2}$ parameters
- $U(N)$ can be parametrized by first parametrizing $O(N)$; then one puts in phase factors at appropriate places $\to N^2 - \frac{N(N-1)}{2}$ phase factors

  Example: $N = 2 \longleftrightarrow U(2)$

$$O = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \in O(2)$$

$$U = \begin{pmatrix} \cos\alpha\, e^{i\phi_1} & \sin\alpha\, e^{i\phi_2} \\ -\sin\alpha\, e^{i\phi_3} & \cos\alpha\, e^{i\phi_4} \end{pmatrix} \text{Ansatz}$$

  Condition that $U \in U(2)$: $\phi_1 - \phi_2 = \phi_3 - \phi_4 \quad (+n\pi)$

  From this equation we see that 3 phases are independent. This agrees with the general formula: $2^2 - \frac{2\cdot 1}{2}$.

- But not all of these phases are physical! (As we will see when looking at the quark mass terms.)

- Mass term for quarks:

$$\bar{U}'_L M^U U'_R + \bar{D}'_L M^D D'_R + h.c.$$

  M^U, M^d diagonal, ≥ 0. M^U and M^d stay invariant, if one redefines the fields according to

$$U'_L = P_u U''_L\,, \qquad U'_R = P_u U''_R\,, \qquad D'_L = P_d D''_L\,, \qquad D'_R = P_d D''_R\,,$$

  provided that P_d and P_u are diagonal "phase matrices":

$$P_d = \begin{pmatrix} e^{i\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\alpha_N} \end{pmatrix}, \qquad P_u = \begin{pmatrix} e^{i\beta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\beta_N} \end{pmatrix}$$

- In the new basis the charged current J_μ^{CC} reads:

$$J_\mu^{CC} = -\frac{g}{\sqrt{2}} \bar{D}''_L \gamma_\mu P_d^\dagger V^\dagger P_u U''_L$$

$$\text{i.e.} V_{ij} \to (P_u^\dagger V P_d)_{ij} = V_{ij} e^{-i(\beta_i - \alpha_j)}$$

One can choose α_i and β_i such that the largest possible amount of phases in V gets removed.

<u>$N = 2$</u>:

Can choose $\beta_1 - \alpha_1$, $\beta_1 - \alpha_2$, $\beta_2 - \alpha_1$

$\beta_2 - \alpha_2 = (\beta_1 - \alpha_2) - (\beta_1 - \alpha_1) + (\beta_2 - \alpha_1)$ is then given.

$\to$ Can eliminate 3 phases. Therefore, in the case $N = 2$ all three phases can be rotated away. No physical phases are remaining!

$\Rightarrow$ no CP-violation is possible in $N = 2$.

<u>N general:</u> $2N - 1$ can be transformed away.

$$\text{Number of physical phases} = N^2 - \frac{N(N-1)}{2} - (2N-1) = \frac{(N-1)(N-2)}{2}.$$

<u>$N = 3$:</u> One physical phase in the CKM matrix V. This phase is the source for CP-violation in the SM.

- <u>Satement:</u> Assume we have calculated the amplitude $\mathcal{A}$ for a specific decay process. The amplitude $\bar{\mathcal{A}}$ for the corresponding CP-conjugated process is then obtained by complex conjugating the CKM elements in $\mathcal{A}$.

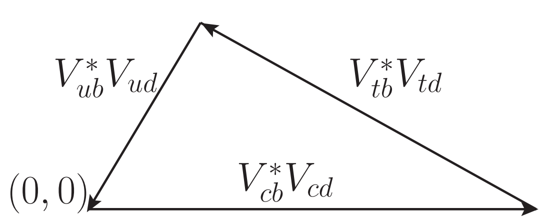
Unitarity triangles

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \in U(3)\,; \quad \text{CKM matrix}$$

Unitarity relation between 1. and 3. column:

$$V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0$$

Representation of this relation in the complex plane: <u>triangle</u>



Phase transformations P_u, P_d rotate the triangle as a whole. The area stays invariant.

<u>Exercise:</u> Show this!

Area J:

$$\vec{a} = (Re(V_{ub}^* V_{ud}); \mathrm{Im}(V_{ub}^* V_{ud}); 0)$$
$$\vec{b} = (Re(V_{cb}^* V_{cd}); \mathrm{Im}(V_{cb}^* V_{cd}); 0)$$
$$2J = |\vec{a} \times \vec{b}| = |\mathrm{Im}(V_{ub} V_{ud}^* V_{cb}^* V_{cd})|$$

Exercise: Show the final expression. Show that it is invariant under phase transformations.

There are 5 additional unitarity triangles. E.g. the one obtained from 2. and 3. column.

$$V_{ub}^* V_{us} + V_{cb}^* V_{cs} + V_{tb}^* V_{ts} = 0$$

Area: $2\ J' = |\mathrm{Im}(V_{ub} V_{us}^* V_{cb}^* V_{cs})|$

We just will show that $J' = J$:

Reason:

$$\begin{aligned}
&\mathrm{Im}(V_{ub} V_{ud}^* V_{cb}^* V_{cd}) \ + \\
&\mathrm{Im}(V_{ub} V_{us}^* V_{cb}^* V_{cs}) \ + \\
&\mathrm{Im}(\underbrace{V_{ub} V_{ub}^* V_{cb}^* V_{cb}}_{\text{real}}) = 0
\end{aligned}$$

Analogously one can show that all 6 triangles have the same area J: This is a measure of CP-violation.

All triangles, except the one formed with 1. and 3. column are strongly degenerate (to a line). The most important one is therefore the one from the 1. and 3. column.

2.3.9 Parametrizations of the CKM Matrix V

Standard parametrization of PDG:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - s_{23}c_{12}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $s_{ij} = \sin(\theta_{ij})$ and $c_{ij} = \cos(\theta_{ij})$. This matrix is exactly unitary.

It turn out that the s_{ij} are relatively small numbers.

$\Rightarrow c_{ij}$ near 1; all θ_{ij} can be chosen in the first quadrant.

Hierarchy: $\quad s_{13} \ll s_{23} \ll s_{12} \ll 1$

A useful parametrization, where one can immediately see the sizes of the various V_{ij}-elements, is the **Wolfenstein-parametrization**:

Insert

$$s_{12} = \lambda, s_{23} = A\lambda^2, s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)$$

into the standard parametrization. By construction, V is still exactly unitary when expressed in terms of A, λ , ρ and η.

Now expand in the small parameter λ ($\lambda \approx 0.2$), obtaining

$$V_{\text{Wolfenstein}} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

Consider again the unitarity triangle (1. + 3. column).

Devide the relation

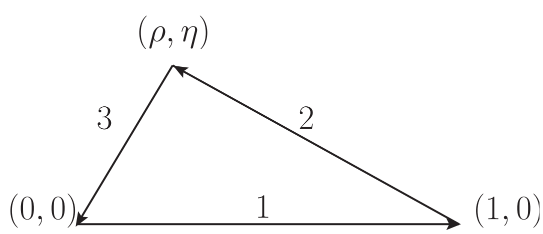
$$V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0$$

by $V_{cb}^* V_{cd}$:

$$\Rightarrow \underbrace{\frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}}}_{3} + \underbrace{\frac{V_{cb}^* V_{cd}}{V_{cb}^* V_{cd}}}_{1} + \underbrace{\frac{V_{tb}^* V_{td}}{V_{cb}^* V_{cd}}}_{2} = 0\,.$$

Doing this in the Wolfenstein paramatrization, we get (neglecting the $O(\lambda^4)$ terms)

$$\begin{aligned} \frac{A\lambda^3(\rho+i\eta)}{-A\lambda^3} + 1 + \frac{A\lambda^3(1-\rho-i\eta)}{-A\lambda^3} &\approx 0 \\ -(\rho+i\eta) + 1 - (1-\rho-i\eta) &\approx 0 \end{aligned}$$



The tip of the triangle has coordinates (ρ,η) (approximate statement.)

Modified Wolfenstein: Insert

$$s_{12} = \lambda\,; \quad s_{23} = A\lambda^2\,; \quad s_{13}e^{i\delta} = \frac{A\lambda^3(\bar{\rho}+i\bar{\eta})\sqrt{1-A^2\lambda^4)}}{\sqrt{1-\lambda^2}\,[1-A^2\lambda^4(\bar{\rho}+i\bar{\eta})]}\,.$$

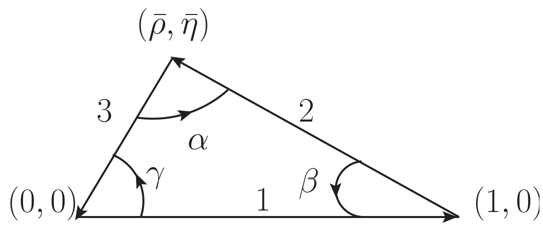
into the standard parametrization.

Exercise: Show that this implies that

$$\bar\rho + i\bar\eta = \frac{-(V_{ud}V_{ub}^*)}{V_{cd}V_{cb}^*}$$

which is phase-convention independent. See PDG!

In the modified version we have



Tip has coordinates $(\bar\rho, \bar\eta)$ (exact statement!)

A large amount of data shows that the CKM picture is realized in nature. Today the fits for A, λ, $\bar\rho$ and $\bar\eta$ are:

PDG 2012	$\lambda = 0.22535 \pm 0.00065$, $A = 0.811^{+0.022}_{-0.012}$ $\bar\rho = 0.131^{+0.026}_{-0.013}$, $\bar\eta = 0.345^{+0.013}_{-0.014}$

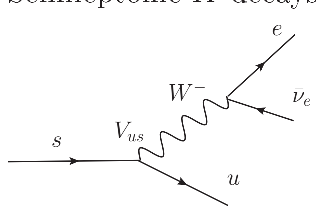
- Measuring decay rates leads often to constraints on the modulus of specific CKM elements: $|V_{ij}|$
- Measuring CP asymmetries often gives direct info on the angles α, β and γ of the unitary triangle.

Strategy: Key-word: "Overconstraining the unitarity triangle". This means: Measure more than necessary and look if everything is consistent.

2.3.10 Measuring some of the CKM elements

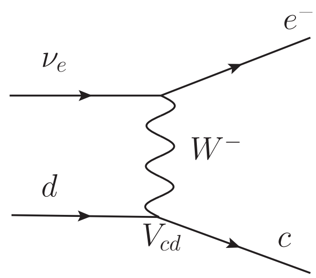
(rather detailed in PDG12; here only brief)

- $|V_{ud}| = 0.97425 \pm 0.00022$ from nuclear β-decay
- Semileptonic K-decays, Hyperon-decays

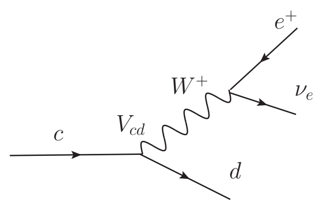


$|V_{us}| = 0.22525 \pm 0.0009$

- Charm production in ν nucleon scattering

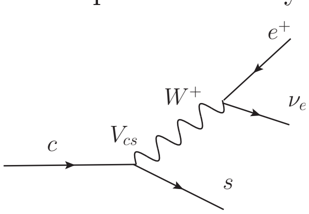


and semileptonic D decays (D mesons contain a charm quark):



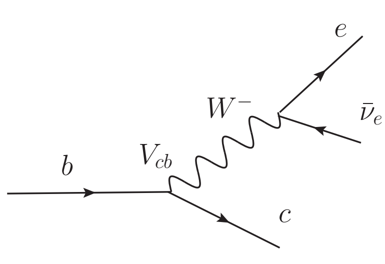
$$|V_{cd}| = 0.230 \pm 0.011$$

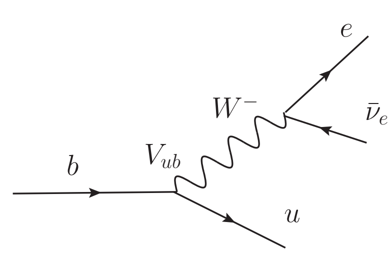
- Semileptonic D decays:



$$|V_{cs}| = 1.006 \pm 0.023$$

- Semileptonic B decays (exclusive and inclusive)

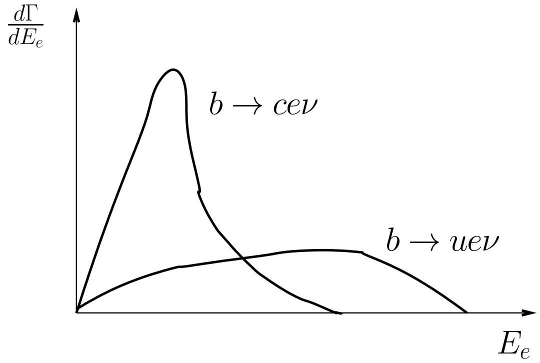




$$|V_{cb}| = (40.9 \pm 1.1) \cdot 10^{-3}\,; \qquad |V_{ub}| = (4.15 \pm 0.49) \cdot 10^{-3}$$

Note that $|V_{ub}| \sim 0.1\,|V_{cb}|$!

Inclusive: only the endpoint spectrum of e^- can be used for the determination of V_{ub}.



$b \to ce\nu$ is much larger.

Only the energy window above the kinematical endpoint from $b \to ce\nu$ can be used for the determination of V_{ub}.

Exclusive: $B \to De\nu$, $B \to D^*e\nu$ for V_{cb} and $B \to \rho e\nu$, $B \to \omega e\nu$ for V_{ub}.

- $t \longrightarrow bW$ and single top production at Fermilab

  average: $|V_{tb}| = 0.89 \pm 0.07$

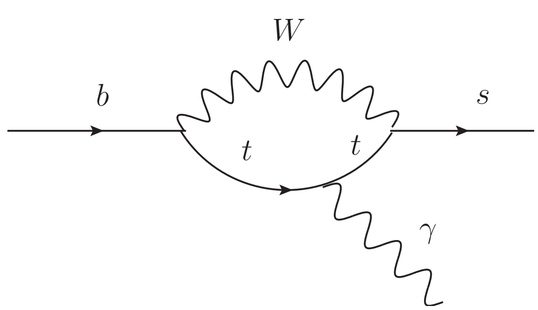
So far, we did not discuss $|V_{td}|$ and $|V_{ts}|$. It is very difficult to extract them from semileptonic top-decays.

- V_{ts} e.g. using CKM unitarity relations and making use of $|V_{cs}| \approx |V_{tb}| \approx 1$. This is of course not a measurement, but it is interesting nevertheless:

$$\underbrace{V_{us}V_{ub}^*}_{\substack{\text{small w.r.t.}\\ 2^{nd}\text{ term}}} + V_{cs}V_{cb}^* + V_{ts}V_{tb}^* = 0$$

$$|V_{cs}V_{cb}^*| \cong |V_{ts}V_{tb}^*| \Rightarrow |V_{ts}| \approx |V_{cb}| = (40.9 \pm 1.1) \cdot 10^{-3}$$

Direct: $b \longrightarrow s\gamma$



$\to |V_{ts}| = (37 \pm 7) \cdot 10^{-3}$ (one of my papers, but not updated.)

Result in agreement with the determination using unitarity relations, but

<u>CAUTION:</u> The process $b \to s\gamma$ is sensitive to new physics! Value given assumes that there is no new physics contribution.

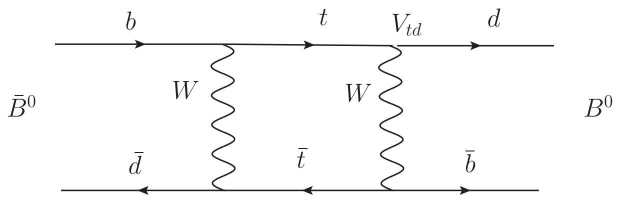
- V_{ts} also from $B_s - \bar{B}_s$ mixing (see discussion in the next point).

  Taking all information on $|V_{ts}|$ together, the PDG gives

$$|V_{ts}| = (42.9 \pm 2.6) \cdot 10^{-3}\,.$$

- $\underline{V_{td}}$ $\bar{B}^0 - B^0$ mixing: B^0 and $\bar{B}^0$ are <u>not</u> eigenstates of $\mathbb{H}_{\text{SM}}$ (w.r.t. $\mathbb{H}_{\text{QCD}}$ they are)

  $\bar{B}^0$ and B^0 can mix (oscillate) into each other. This mixing is induced through second order weak interaction.



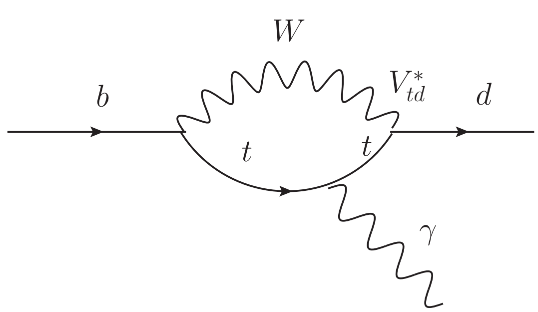
(B^0,$\bar{B}^0$) mixing diagram

B_1^0, B_2^0 : States (with exponential decay law) which diagonalize the mass matrix

$\Delta m = m_{B_1^0} - m_{B_2^0}$ (dominated by the diagram above) is sensitive on $|V_{td}|$!

<u>In analogy:</u> $\bar{B}_s^0 - B_s^0$ mixing is sensitive on $|V_{ts}|$.

$|V_{td}|$ in the future probably also from $b \longrightarrow d\gamma$



$$|V_{td}|^{\text{PDG12}} = (8.4 \pm 0.6) \cdot 10^{-3}$$

3 Inclusive semileptonic B-decays

3.1 General comments on inclusive decays

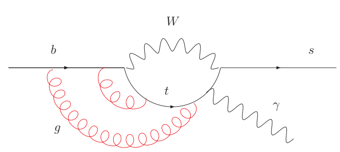
The decay width for inclusive B-meson decays (like $B \to X_s\gamma$) can be written in the following way:

$$\Gamma[B \to X_s\gamma] = \Gamma[b \to s\gamma(g)] + \text{corr. in } \Lambda_{QCD}/m_b\,.$$

- no linear corrections in Λ_{QCD}/m_b
- Corr. start at $\mathcal{O}(\Lambda_{QCD}^2/m_b^2)$; they are related to the motion of the b-quark inside the meson; their numerical effect is of order 4% at the level of the decay width. Perhaps Javier will discuss this.
- We concentrate on the partonic piece, including perturbative QCD corrections.

Well-known: Often, the decay rate is significantly enhanced by QCD-effects.

There are large logs of the form (n gluons exchanged)



$$\left(\frac{\alpha_s}{\pi}\right)^n \log^n \frac{m_b^2}{M^2} \qquad M = m_t, m_W \,:\, \text{leading logs (LL)}$$

$$\left(\frac{\alpha_s}{\pi}\right)^n \log^{n-1} \frac{m_b^2}{M^2} \qquad \text{next-to-leading logs (NLL)}$$

To get a reasonable result, one has to resum at least the LL and NLL terms.

Useful machinery to achieve this resummation: Construct an effective Hamiltonian and resum logs using renormalization group equation (RGE) techniques. This is what we want to learn in the first part of this course.

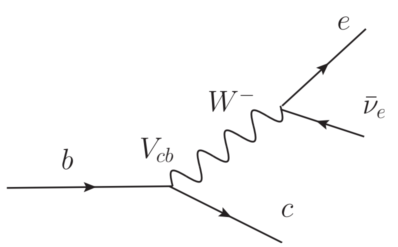
The most reliably calculable B-decays are the so-called inclusive semileptonic decays:

$$b \longrightarrow c\, e\, \bar{\nu}\,; \quad b \longrightarrow u\, e\, \bar{\nu}\,.$$

In this case, there are no large logs; consequently, they can be calculated in fixed order perturbation theory.

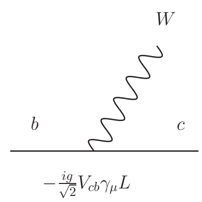
We first give the lowest order results and then discuss the effects of virtual- and bremsstrahlung QCD corrections at order α_s^1.

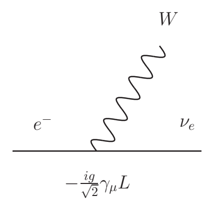
3.2 $\Gamma(b \to c\, e\, \nu)$ and $\Gamma(b \to u\, e\, \nu)$ at lowest order



lowest order diagram

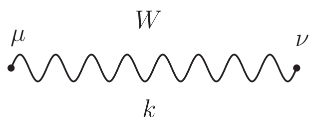
Vertices:





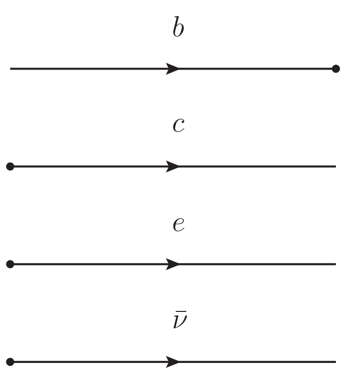
$$L \equiv \frac{1-\gamma_5}{2}$$

Propagator:



$$\frac{1}{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}\right)\frac{1}{k^2 - m_W^2} \qquad \text{(in unitary gauge)}$$

Incoming/Outgoing fermions:



$u(p_b)$

$\bar{u}(p_c)$

$\bar{u}(p_e)$

$v(p_\nu)$

Dirac equation:

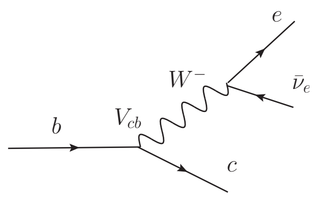
$$\not{p}\, u(p) = m\, u(p)$$

$$\not{p}\, v(p) = -m\, v(p)$$

$$S = 1 + iT\,; \qquad \text{decomposition of the scattering matrix}$$

The decay amplitude M then reads

$$M = \quad \langle c\, e^- \,\bar{\nu} | T | b \rangle = (2\pi)^4\, \delta^4(p_b - p_c - p_e - p_\nu)\, \bar{u}(p_c)\gamma_\mu L\, u(p_b)\, \bar{u}(p_e)\gamma_\nu L\, v(p_\nu) \times \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_W^2}\right)\frac{1}{k^2 - m_W^2}\left(-\frac{ig}{\sqrt{2}}\right)^2 \frac{1}{i} V_{cb}$$

- 

  $$k = O(m_b) \quad \to \frac{1}{k^2 - m_W^2} \approx -\frac{1}{m_W^2}$$

  This is what makes the weak interaction weak in low energy processes, **not** the weak coupling g ($g = O(e)$!).

- Term $\sim \frac{k^\mu k^\nu}{m_W^2}$ in the propagator: Can also be neglected.

- We put $m_e = 0$ in the following, obtaining

  $$\begin{aligned} M &= (2\pi)^4 \delta^4(p_b - p_c - p_e - p_\nu)\, t \\ t &= \bar{u}(p_c)\gamma_\mu L\, u(p_b) \bar{u}(p_e)\gamma^\mu L\, v(p_\nu) \left(-\frac{ig^2}{2m_W^2} V_{cb}\right) \end{aligned}$$

  t is sometimes called the "reduced matrix element".

$$t = \bar{u}(p_c)\gamma_\mu L\, u(p_b) \bar{u}(p_e)\gamma^\mu L\, v(p_\nu) \underbrace{\left(\frac{-ig^2}{2m_W^2} V_{cb}\right)}_{\equiv V}$$

Decay rate:

$$d\Gamma = \frac{1}{2m_b} |t|^2_\Sigma (2\pi)^4 \delta^4(p_p - p_c - p_e - p_\nu) d\mu(p_c) d\mu(p_e) d\mu(p_\nu) \tag{3.1}$$

$$d\mu(p) \equiv \frac{d^3p}{(2\pi)^3 2p^0}, \quad p^0 = \sqrt{m^2 + \vec{p}^2}$$

Exercise:

$$d\mu(p) = \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0)$$

$|t|^2_\Sigma$: The index Σ means to sum over spins, colours in final state and to average over spins, colours in initial state.

After some pages of calculation one arrives at

$$\frac{d\Gamma}{dq^2} = \frac{64 G_F^2}{192\pi^2} |V_{cb}|^2 \left[2(p_b q)(p_c q) + q^2 p_b p_c\right] \frac{\left[(m_b^2 - (m_c + \sqrt{q^2})^2)(m_b^2 - (m_c - \sqrt{q^2})^2)\right]^{1/2}}{16\pi m_b^3}.$$

This is called the "Distribution of the invariant mass squared of the lepton pair ($q^2 = (p_e + p_\nu)^2$)". q^2 varies in the range $0 \le q^2 \le (m_b - m_c^2)$. The scalar products in this epression are understood to be

$$p_b p_c = \frac{m_b^2 + m_c^2 - q^2}{2}\,; \qquad p_b q = \frac{m_b^2 - m_c^2 + q^2}{2}\,; \qquad p_c q = \frac{m_b^2 - m_c^2 - q^2}{2}\,.$$

To get the (total) decay width, we have to integrate over q^2, obtaining

$$\Gamma = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} g\left(\frac{m_c}{m_b}\right)\,; \qquad (b \to c e \bar{\nu}_e)$$
$$g(z) = 1 - 8z^2 + 8z^6 - z^8 - 24 z^4 \ln z$$

$g(z)$ is called "phase-space function".

Γ shows strong dependence on m_b: m_b^5! An uncertainty in m_b has a strong influence on Γ!

Many other B-decays also involve m_b^5, e.g. $\Gamma(b \to s\gamma)$.

One therefore often normalizes with the semileptonic expression, e.g.:

$$BR(b \to s\gamma) = \left(\frac{\Gamma(b \to s\gamma)}{\Gamma(b \to c e \bar{\nu}_e)}\right)_{th} \underbrace{\left(\frac{\Gamma(b \to c e \bar{\nu}_e)}{\Gamma_{tot}}\right)_{exp.}}_{(10.49 \pm 0.46)\%}$$

Phase-space function:

$$\frac{m_c}{m_b} = 0.29 \pm 0.02 \quad \longrightarrow g\left(\frac{m_c}{m_b}\right) \approx 0.54$$

It would be a bad approximation to neglect m_c with respect to m_b!

$b \to u e \bar{\nu}_e$:

There one can safely put $m_u = 0$ and gets

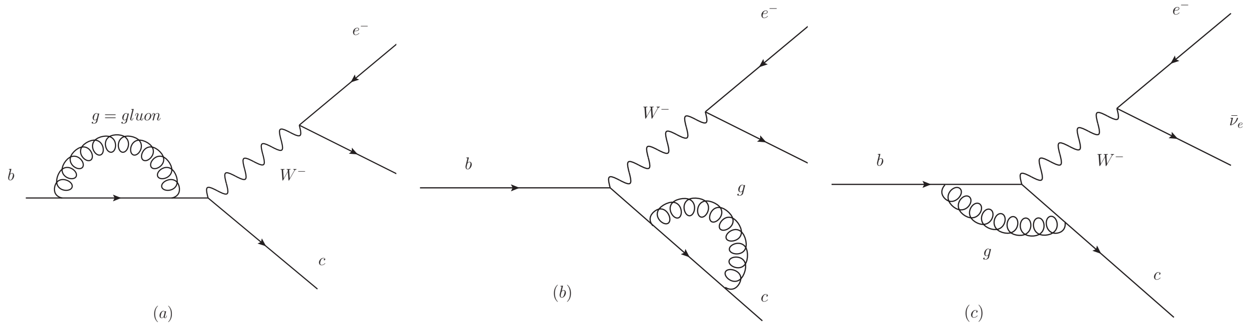
$$\Gamma_{b \to u e \bar{\nu}_e} = \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3}\,.$$

3.3 QCD Corrections to $b \to c\,e\,\bar{\nu}_e$, $b \to \, u\,e\,\bar{\nu}_e$

There are so-called virtual gluon corrections and gluon Bremsstrahlung corrections which have to be combined. I do not present the calculations which in principle would be quite interesting, because they involve UV renormalization and also how to deal properly with infrared singularities. Please ask me for my lecture notes of a more detailed course when you are interested in the details of these calculations.

3.3.1 Virtual gluon corrections

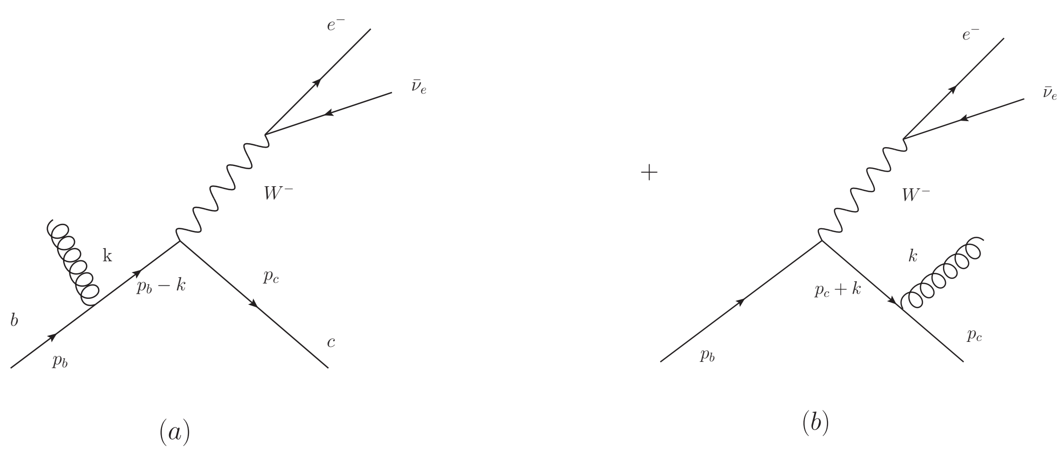
The bare diagrams are:



The UV-renormalized contribution to the decays width suffers from infrared singularities.

3.3.2 Bremsstrahlung process $b \to ce\nu + g$

The contributing diagrams are:



"Only" tree-graphs, but the <u>four</u> particles in the final state lead to rather complicated phase-space integrals. $\to$ Not a trivial problem at all!

Also this contribution to the decay width suffers from infrared singularites.

When combining virtual- and Bremsstrahlung corrections, these singulaties cancel, leading to a physical result.

3.3.3 Semipleptonic decay width in $\mathcal{O}(\alpha_s)$ QCD

For the exact result see e.g. in Y.Nir, PLB221(1989)84. A good approximation the semileptonic decay to order α_s in QCD reads

$$\Gamma(b \to c\, e^- \bar{\nu} + c\, e^- \bar{\nu} g) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} g\left(\frac{m_c}{m_b}\right)\left[1 - \frac{2\alpha_s}{3\pi} f\left(\frac{m_c}{m_b}\right)\right],$$

where the functions g and f read

$$\begin{aligned} g(u) &= 1 - 8u^2 + 8u^6 - u^8 - 24u^4 \ln u; \qquad \alpha_s \equiv \frac{g_s^2}{4\pi}, \\ f(u) &= \left(\pi^2 - \frac{31}{4}\right)(1-u)^2 + \frac{3}{2}. \end{aligned}$$

m_b and m_c are understood to be <u>pole-masses</u>; this is important to specify, because otherwise the explicit expression for f would be different.

Numerically we have:

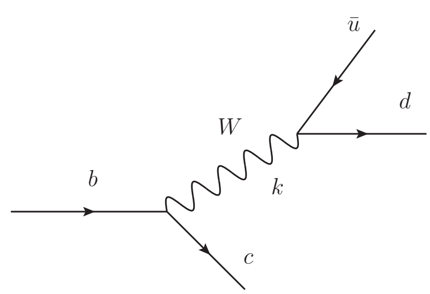
$$\frac{m_c}{m_b} \approx 0.3; \quad \alpha_s \approx 0.2; \quad \left[1 - \frac{2\alpha_s}{3\pi} f\left(\frac{m_c}{m_b}\right)\right] \approx 0.89.$$

$\to$ QCD corrections reduce the semileptonic decay width by about $\approx 11\%$.

4 Effective Hamiltonian for $b \to c\,\bar{u}\,d$

B-meson decays are induced through the weak interaction. In the theoretical description (in the SM) W- and Z-propagators occur; also the top-quark can contribute in loops. The decay amplitudes therefore depend on mass-scales which are very different: On one hand on m_b and on the other hand on m_W, m_t, m_Z. It is possible to construct an equivalent theory in which only the "light" fields are present: b, c, s, d, u; τ, μ, e, ν_e, ν_μ, ν_τ; γ, g. We consider in the following the decay $b \to c\,\bar{u}\,d$ for which we can explicitly illustrate the essential features of the construction of the effective theory.

4.1 Tree-level



$$A = \left(-\frac{ig}{\sqrt{2}}\right)^2 (\bar{d}\gamma_\mu L\,u)\frac{1}{i}\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{m_W^2}}{k^2 - m_W^2}(\bar{c}\gamma_\nu L\,b)V_{cb}V_{ud}^*$$

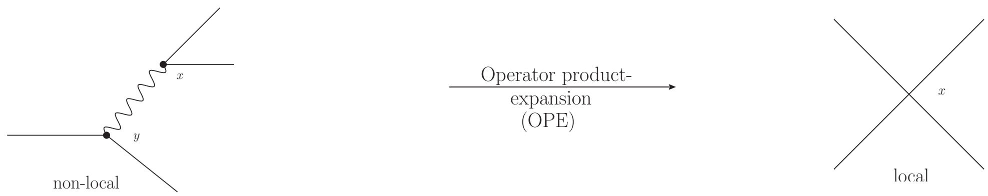
$k = \mathcal{O}(m_b) \ll m_W \;\Rightarrow$ can expand the propagator in the momentum transfer k.

$$\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{m_W^2}}{k^2 - m_W^2} = -\frac{g^{\mu\nu}}{m_W^2} + \mathcal{O}\left(\frac{k^2}{m_W^4}\right)$$

The amplitude can then be written as

$$A = \frac{1}{i}\frac{g^2}{2}\left[\frac{1}{m_W^2}(\bar{d}\gamma_\mu L\,u)(\bar{c}\gamma^\mu Lb)] + \frac{1}{m_W^4}\,(\text{dim 8 Operators}) + \frac{1}{m_W^6}\,(\text{dim 10 Operators}) + ...\right] V_{cb}V_{ud}^*$$

The dimension 8, 10, ... operators contain the same fields as the dimension 6 operator. In momentum space they contain powers of k, i.e., powers of external momenta. In position space these momenta correspond to derivatives. All these operators are local operators.



The amplitude is dominated by the operator(s) of lowest dimension. Higher-dimensional operators are suppressed by inverse powers of m_W. We only will need the contribution(s) of the leading operator(s), i.e., with lowest dimension. From the amplitude A we can immediately read off the effective Hamiltonian.

$$H_{\text{eff}}^{\text{dim }6} = \underbrace{\frac{g^2}{2} V_{cb} V_{ud}^* \frac{1}{m_W^2}}_{\text{prefactor}} \underbrace{(\bar{d}\gamma_\mu L\, u)(\bar{c}\gamma^\mu L\, b)}_{\text{operator}}$$

The prefactor is determined in such a way that

$$\langle c\bar{u}d| - iH_{\text{eff}}|b\rangle = A\,.$$

g^2 is often written in terms of G_F; the connection is

$$g^2 = 4\sqrt{2}G_F m_W^2 \leftarrow \text{defines } G_F\,.$$

$$\boxed{H_{\text{eff}}^{\text{dim }6} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* C_2 O_2}$$

$$O_2 = (\bar{d}\gamma_\mu L\, u)(\bar{c}\gamma^\mu L\, b) \quad \leftarrow \text{is simply called } O_2 \text{ in literature}$$
$$C_2 = 1 \quad \leftarrow \text{Wilson coefficient}$$

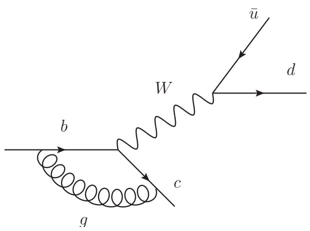
Remark: We "contracted-out" the W-field in the sense of the Wick-theorem. We then expand the corresponding propagator. The analogous thing in the path-integral formalism would be: Integrate over the W-field. After this step a non-local action results. The subsequent OPE then leads to the same local H_{eff}. See Buras hep-ph/9806471, p.53.

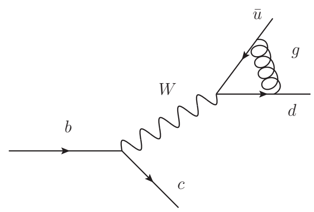
Remark: We will see that the Wilson coefficient $C_2 = 1$ gets modified by QCD effects.

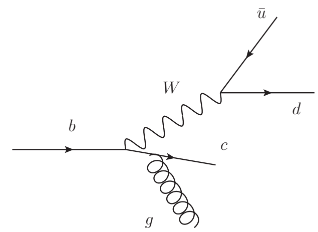
4.2 QCD-effects; principle comments

We now want to consider QCD corrections of order $\alpha_s = \frac{g_s^2}{4\pi}$. Is it still possible to expand the W-propagator?

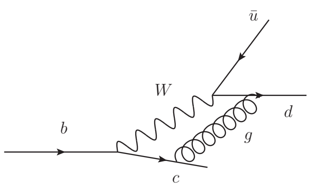
- no problem in the diagrams





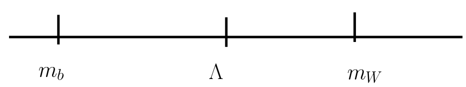


- not so clear in the diagrams as e.g.



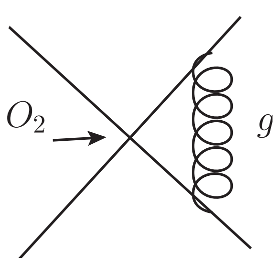
Depending on the value of the loop-momentum, large or small momenta flow through the W-line: Split loop-momentum into two regions:

$$(I) \quad 0 < |l_\alpha| < \Lambda$$
$$(II) \quad \Lambda < |l_\alpha| < \infty$$



Choose Λ such that $m_b \ll \Lambda \ll m_W$.

Region (I): Can expand the propagator; we only keep the leading term in $1/m_W^2$. The calculation then boils down to working out the following diagram:



$|l_\alpha| < \Lambda$

Region (II): External momenta are small compared to the m_W-mass and the loop-momentum. One therefore can expand in the external momenta. Leading term corresponds to p_i=0. Higher powers in p_i are suppressed by $(\frac{p_i}{m_W})^2 \leftrightarrow$ higher-dimensional operators. Therefore we can simply put $p_i = 0$ in this region.

$\rightarrow$ The momentum dependence of the decay amplitude comes from region (I), i.e., from the diagram shown there.

$\rightarrow$ Non-trivial m_W-dependence, which is <u>not</u> of the form $1/m_W^n$, comes from region (II). This one manifests itself in terms of the form $\ln \frac{m_W}{\Lambda}$ as we will see.

The <u>result</u> of the QCD corrected decay amplitude $b \to c\bar{u}d$ can be written as

$$A = \langle c\bar{u}d| - iH_{\text{eff}}^{\text{dim } 6}|b\rangle$$

$$\begin{aligned} H_{\text{eff}}^{\text{dim } 6} &= \frac{4G_F}{\sqrt{2}} V_{cb}V_{ud}^*[C_1(\Lambda)O_1(\Lambda) + C_2(\Lambda)O_2(\Lambda)] \\ O_1 &= (\bar{d}_\alpha \gamma_\mu L\, u_\beta)(\bar{c}_\beta \gamma^\mu L\, b_\alpha) \quad \alpha, \beta : \text{colour indices} \\ O_2 &= (\bar{d}_\alpha \gamma_\mu L\, u_\alpha)(\bar{c}_\beta \gamma^\mu L\, b_\beta) \end{aligned}$$

$O_1(\Lambda)$, $O_2(\Lambda)$ means that one cuts off the loop momentum at Λ when calculating their matrix-elements. The matrix-elements $\langle c\bar{u}d|O_1(\Lambda)|b\rangle$ and $\langle c\bar{u}d|O_2(\Lambda)|b\rangle$ completely contain the physics of the small momenta, in particular the dependence on the external momenta. The Wilson coefficients contain the dependence of the masses of the heavy particles which are integrated out (here: the W-boson).

$$A \sim \underbrace{C_2(\Lambda)}_{(C_2^0 + C_2^1 \alpha_s)} \underbrace{\langle O_2(\Lambda)\rangle}_{(M_2^0 + M_2^1 \alpha_s) +} + \underbrace{C_1(\Lambda)}_{(C_1^0 + C_1^1 \alpha_s)} \underbrace{\langle O_1(\Lambda)\rangle}_{(M_1^0 + M_1^1 \alpha_s) + \mathcal{O}(\alpha_s^2)}$$

In practice one fixes the Wilson-coefficients in such a way that one calculates the decay matrix element (or the corresponding Greens-function) in both, in the full theory and in the effective theory. In the full theory one integrates over all loop-momenta. In the effective theory only over those with $|l_\alpha| < \Lambda$.

The Wilson coefficients can then be fixed by requiring that the results are the same. This "fixing" is called <u>matching</u>.

Suppose we only want to fix the Wilson coefficients, i.e., we are interested in the matching calculation only. As the Wilson coefficients do <u>not</u> depend on the external momenta, the <u>matching</u> calculation can be done for a suitable configuration of external momenta, for which the calculation becomes as simple as possible.

<u>Remark</u>: The explicit cutting-off of loop-momenta is convenient to understand the principle. Technically, however, this procedure is very cumbersome. Instead one can integrate in the effective theory over all loop-momenta (just using the <u>dimensionally regularized version</u>). The dependence on the external momenta is the same. The difference is in momentum-independent terms. This means the that Wilson coefficients $C_i^{\text{dim. reg}}$ and $C_i(\Lambda)$ differ by those terms. But the product $C_i\langle O_i\rangle$ is the same in both schemes.

$\rightarrow$ The product $C_i\langle O_i\rangle$ is the physical object, not C_i and $\langle O_i\rangle$ individually.

Remark: In dimensional regularization the "renormalization scale" μ (or $\bar{\mu} = \mu\, \mathrm{e}^{\gamma_E/2}/\sqrt{4\pi}$) occurs. It turns out that the logarithmic dependence on μ in the dimensional result is the same as the Λ-dependence in the cut-off procedure. One therefore can say that μ also serves as the separation scale between large and small loop momenta.

Exercise:

$$
\begin{aligned}
I_1 &\doteq \bar{\mu}^{2\epsilon} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{[\vec{l}_E^{\,2} + C]^2} \\
I_2 &\doteq \int_{|\vec{l}_E|<\Lambda} \frac{d^4 l_E}{(2\pi)^4} \frac{1}{[\vec{l}_E^{\,2} + C]^2}
\end{aligned}
$$

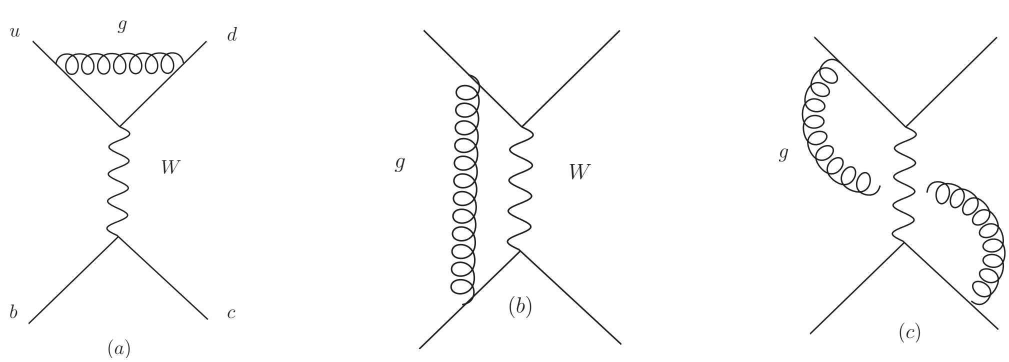
Show that the coefficient of $\ln\Lambda$ in I_2 and the coefficient of $\ln\bar{\mu}$ in I_1 coincide.

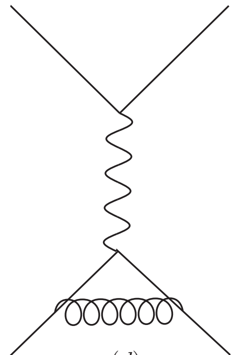
In the following chapter we will explicitly do the matching calculation, i.e. the extraction of the Wilson-coefficients $C_1(\mu)$, $C_2(\mu)$ at order $\alpha_s = g_s^2/4\pi$ in QCD.

4.3 Matching of H_{eff} for $b \rightarrow c\bar{u}d$ at $\mathcal{O}(\alpha_s)$

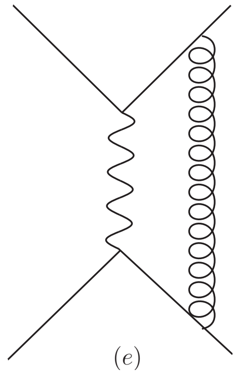
$$
\begin{aligned}
H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} & V_{cb}V_{ud}^{*}[C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)]\,, \\
& O_1 = (\bar{d}_\alpha\gamma_\mu L\, u_\beta)(\bar{c}_\beta\gamma^\mu L\, b_\alpha)\,, \qquad O_2 = (\bar{d}_\alpha\gamma_\mu L\, u_\alpha)(\bar{c}_\beta\gamma^\mu L\, b_\beta)\,, \\
& C_1(\mu) = 0 + ?\, \alpha_s\,, \qquad C_2(\mu) = 1 + ?\alpha_s\,.
\end{aligned}
$$

a) Full theory

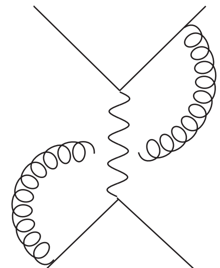




(d)



(e)



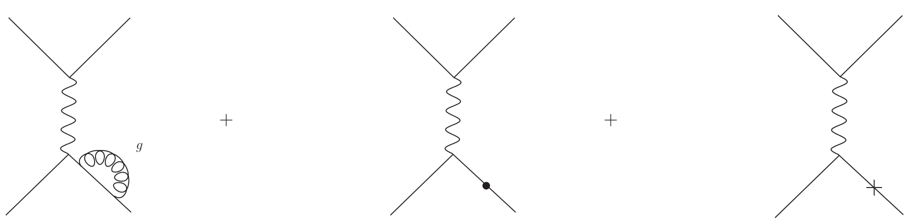
(f)

As the Wilson coefficients only depend on the masses of the heavy particles (and μ), we choose the quark masses to be zero and the external momenta identical (all $= p$), with $p^2 < 0$. Disadvantage: One looses of course the explicit check that the Wilson coefficients indeed only depend on the heavy masses.

Why don't we choose $p = 0$? Just to avoid collinear and infrared divergences! We only will keep p^2 as a regulator of such singularities. In terms where p^2 does not serve as a regulator, we will put $p^2 = 0$.

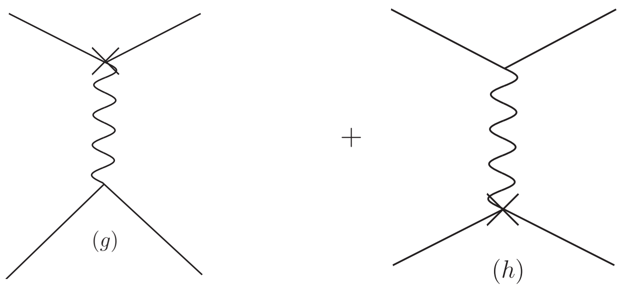
Remark: Actually, we should take into account self-energy diagrams in addition, together with their counterterms.

e.g



These self-energiy contributions appear exactly in the same way when we will calculate the matrix elements of the operators in the effective theory. Therefore they will give no contributions to the Wilson-coefficients. We therefore can omit them (of course in the full and in the effective theory).

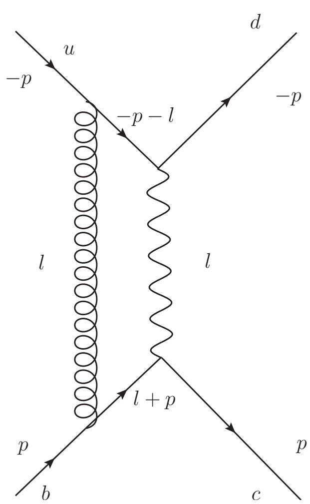
Remark: The counter-terms to the vertex corrections we do take into account:



At the cross in the vertex we have the renormalization factor $(\bar{Z}_1 - 1)$, where $\bar{Z}_1 = \sqrt{\bar{Z}_{2b}\,\bar{Z}_{2c}}\,\bar{Z}_g$ (with $\bar{Z}_g = 1$).

As an example, we consider diagram (b); then we will give the result for the sum of (a)-(f).

<u>Calculation of diagram (b)</u>: $b + u \to d + c$



Box-Diagram
must be UV-finite. Why?

$$A_b^{\text{full}} = \left(-\frac{ig}{\sqrt{2}}\right)^2 (-ig_s)^2 \frac{ii}{ii} \int \frac{d^4 l}{(2\pi)^4} \frac{[\bar{d}\gamma_\mu L(-\not{l} - \not{p})\gamma_\alpha T^A u]\,[\bar{c}\gamma^\mu L(\not{l} + \not{p})\gamma^\alpha T^A b]}{(l+p)^2\,(l+p)^2\,l^2\,(l^2 - m_W^2)}$$

$T^A = \frac{\lambda^A}{2}$ = colour matrix

CKM factors omitted; they will be restored in the final result.

<u>Feynman-parametrization:</u>

$$\frac{1}{a^2 bc} = \int_0^1 dy\,dx\,6y^2 \frac{x}{[axy + by(1-x) + c(1-y)]^4}$$

$$\begin{array}{ll|l} a = (l+p)^2 = l^2 + 2lp + p^2 & & xy \\ b = l^2 - m_W^2 = l^2 - m_W^2 & & y(1-x) \\ c = l^2 \qquad\quad = l^2 & & (1-y) \end{array}$$

$$
\begin{aligned}
axy + by(1-x) + c(1-y) =& l^2 + 2lpxy + p^2xy - m_W^2 y(1-x) \\
=& \underline{(l+pxy)^2 - C} \\
& C = m_W^2 y(1-x) - p^2xy + p^2x^2y^2
\end{aligned}
$$

- $l \to l - pxy$ shift. Then put $p = 0$ in the numerator, because $p \neq 0$ only used as regulator.

- $\not{l}...\not{l} \overset{\int}{=} \gamma_\beta ... \gamma^\beta \frac{l^2}{4}$

  $\to A_b^{\text{full}} = \frac{-g^2 g_s^2}{2} (\bar{d}\gamma_\mu L \gamma_\beta \gamma_\alpha T^A u)(\bar{c}\gamma^\mu L \gamma^\beta \gamma^\alpha T^A b) \frac{1}{4} \int 6y^2 x \frac{d^4 l}{(2\pi)^4} \frac{l^2}{[l^2 - C]^4}$

- $\gamma_\mu L \gamma_\beta \gamma_\alpha \otimes \gamma^\mu L \gamma^\beta \gamma^\alpha \equiv 16 \gamma_\mu L \otimes \gamma^\mu L$ identity (at least in $d = 4$)

$$
A_b^{\text{full}} = -12 g^2 g_s^2 (\bar{d}\gamma_\mu L T^A u)(\bar{c}\gamma^\mu L T^A b) \int y^2 x \frac{d^4 l}{(2\pi)^4} \frac{l^2}{[l^2 - C]^4} dx\, dy
$$

- $(\bar{d}_\alpha \gamma_\mu L T^A_{\alpha\beta} u_\beta)(\bar{c}_\gamma \gamma^\mu L T^A_{\gamma\delta} b_\delta) = -\frac{1}{6}(\bar{d}_\alpha \gamma_\mu L\, u_\alpha)(\bar{c}_\beta \gamma^\mu L\, b_\beta) + \frac{1}{2}(\bar{d}_\alpha \gamma_\mu L\, u_\beta)(\bar{c}_\beta \gamma^\mu L\, b_\alpha)$

  <u>Used:</u> $T^A_{\alpha\beta} T^A_{\gamma\delta} = -\frac{1}{2N_c} \delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}$

$$
A_b^{\text{full}} = -12 g^2 g_s^2 \left[\frac{1}{2} S_1 - \frac{1}{6} S_2 \right] f
$$

$$
\begin{aligned}
& S_1 \equiv \langle O_1 \rangle_{\text{tree}} \\
& S_2 \equiv \langle O_2 \rangle_{\text{tree}} \\
& f = \int y^2 x \frac{d^4 l}{(2\pi)^4} \frac{l^2}{[l^2 - C]^4} dx\, dy \\
& \quad C = m_W^2 y(1-x) - p^2 xy(1-xy) \\
& f = \int dx\, dy y^2 x \left(\frac{-i}{48\pi^2} \right) \frac{1}{C} \\
& f = -\frac{i}{48\pi^2} \int dx\, dy \frac{xy}{m_W^2(1-x) - p^2 x(1-xy)}
\end{aligned}
$$

For $p^2 = 0$ the integral would be divergent at $x = 1$.

$$
f = \frac{i}{96\pi^2} \frac{1}{m_W^2} \left[\ln\left(\frac{-p^2}{m_W^2} \right) - \frac{1}{2} \right] + \text{terms which vanish in the limit } p^2 \to 0.
$$

The final result for A_b^{full} reads

$$
A_b^{\text{full}} = -i \frac{4G_F}{\sqrt{2}} \frac{\alpha_s}{4\pi} \left[\frac{1}{2} S_1 - \frac{1}{6} S_2 \right] \left[\ln\left(\frac{-p^2}{m_W^2} \right) - \frac{1}{2} \right] \cdot 4\,.
$$

Sum of diagrams (a)-(f):

$$A_{\text{full}} = -i\frac{4G_F}{\sqrt{2}}V_{cb}V_{ud}^* \left[\left(1 + 2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2} + c_1\right)\right) S_2 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\ln\frac{m_W^2}{-p^2} + c_2\right) S_2 - 3\frac{\alpha_s}{4\pi}\left(\ln\frac{m_W^2}{-p^2} + c_3\right) S_1\right]$$

$S_1 \equiv \langle O_1\rangle_{\text{tree}}$; $S_2 \equiv \langle O_2\rangle_{\text{tree}}$; $C_F = \frac{4}{3}$; $N_c = 3$; c_1, c_2, c_3 are numbers.

1. line: Diagrams (a)+(d)

2. line: (b)+(c)+(e)+(f)

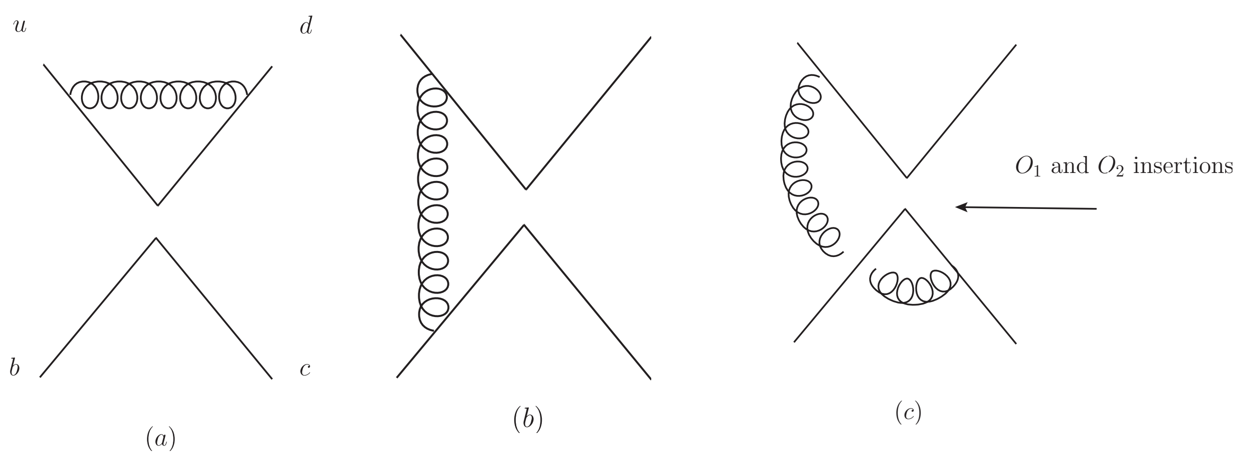
Effect of counterterms (g)+(h): They exactly cancel the $1/\epsilon$ in A_{full} above.

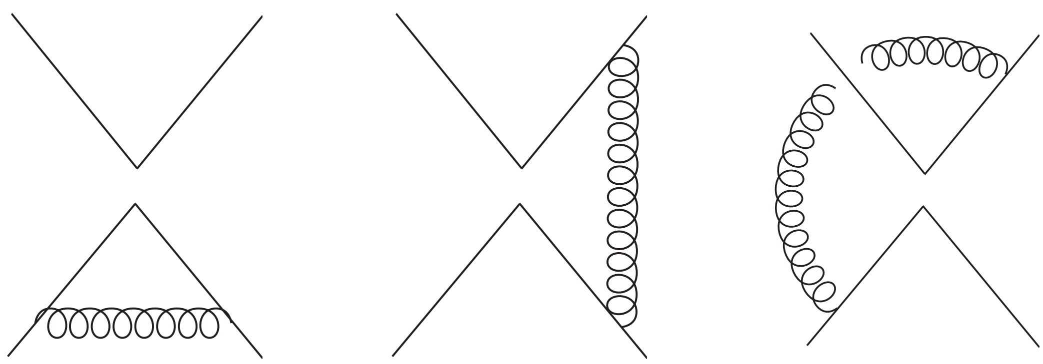
For the renormalized result in the full theory we get

$$A_{\text{full}}^{\text{ren}} = -i\frac{4G_F}{\sqrt{2}}V_{cb}V_{ud}^* \left[\left(1 + 2C_F\frac{\alpha_s}{4\pi}\left(\ln\frac{\mu^2}{-p^2} + c_1\right)\right) S_2 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\ln\frac{m_W^2}{-p^2} + c_2\right) S_2 -3\frac{\alpha_s}{4\pi}\left(\ln\frac{m_W^2}{-p^2} + c_3\right) S_1\right].$$

We now calculate the same amplitude in the effective version of the theory.

b) effective theory





(d) (e) (f)

For illustration, we work out again diagram (b) with O_1 and O_2 inserted. We anticipate that there will be divergences and therefore work in d-dimensions.

$$\langle O_2\rangle_b = ii\frac{1}{i}(-ig_s)^2\bar{\mu}^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{(\bar{d}\gamma_\mu L(-\not{l}-\not{p})\gamma_\alpha T^A u)(\bar{c}\gamma^\mu L(\not{l}+\not{p})\gamma^\alpha T^A b)}{(l+p)^2\,(l+p)^2\,l^2}$$

$$\frac{1}{a^2b} = \int_0^1 dx\frac{2x}{[ax+b(1-x)]^3}$$

$$\begin{array}{ll|l} a=(l+p)^2 & = l^2+2lp+p^2 & x \\ b=l^2 & = l^2 & (1-x) \end{array}$$

$$[ax+b(1-x)] = l^2+2lpx+p^2x = (l+px)^2 - C$$
$$C = -p^2x(1-x)$$

Shift $l\to l-px$; again we omit $\not{p}$ in the numerator, because we are interested in the limit $p\to 0$.

$$\langle O_2\rangle_b = ig_s^2\bar{\mu}^{2\epsilon}\frac{1}{d}\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2-C]^3}\,2\,x\,dx\,\underbrace{(\bar{d}\gamma_\mu L\gamma_\beta\gamma_\alpha T^A u)(\bar{c}\gamma^\mu L\gamma^\beta\gamma^\alpha T^A b)}_{16(\bar{d}\gamma_\mu L T^A u)(\bar{c}\gamma^\mu L T^A b)}$$

To get the term under the braces, we used a 4-dimensional identity. When applying it, one correctly gets only the $1/\epsilon$-pole and the term $\log\mu$. The other terms, however, come out wrong. How to reduce

$$\gamma_\mu L\gamma_\beta\gamma_\alpha \otimes \gamma^\mu L\gamma^\beta\gamma^\alpha$$

in d dimensions, depends on the choice of the so-called <u>evanescent operators</u>. We will see, that one needs to a good approximation – the so-called <u>leading logarithmic approximation</u> – only the $\log\mu$ terms.

Only when going beyond leading-log, the business with evanescent operators becomes relevant. So we get:

$$\langle O_2\rangle_b = ig_s^2\frac{1}{4}\underbrace{\bar\mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2-C]^3}\,2\,x\,dx}_{\frac{i}{16\pi^2}[\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}+?]}\,16\left[\frac{1}{2}S_1-\frac{1}{6}S_2\right]$$

$$\langle O_2\rangle_b = -\frac{g_s^2}{16\pi^2}4\,\left[\frac{1}{2}S_1-\frac{1}{6}S_2\right]\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}+\underbrace{?}_{\substack{\text{depends on}\\\text{choice of}\\\text{ev. operators}}}\right)$$

Calculation of $\langle O_1\rangle_b$ similar. The result of the sum of all diagrams (a)-(f) reads:

$$\langle O_1\rangle^0 = \left(1+2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)\right)S_1+\frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)S_1-3\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)S_2\,,$$
$$\langle O_2\rangle^0 = \left(1+2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)\right)S_2+\frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)S_2-3\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)S_1\,.$$

The index ^{0} means: unrenormalized matrix elements, i.e., really just the contributions from diagrams (a)-(f).

As in the full theory we now do the renormalization of the quark fields (and coupling constants) in the lowest order contribution, i.e., one takes into account the corresponding counterterms.

In our case only the renormalization of the quark-fields is relevant (no other fields or coupling constants)

$$\bar Z_2 = 1-\frac{\alpha_s}{4\pi}C_F\frac{1}{\epsilon}$$

The counterterm contributions then are:

$$\langle O_2\rangle^{\text{c.t.}}_{\text{fields}} = (\bar Z_2^2-1)\langle O_2\rangle^{\text{tree}}\,,$$
$$\langle O_1\rangle^{\text{c.t.}}_{\text{fields}} = \underbrace{(\bar Z_2^2-1)}_{\substack{\text{starts at}\\\text{order }\alpha_s}}\langle O_1\rangle^{\text{tree}}\,.$$

One sees that these counterterms just remove the <u>first</u> $1/\epsilon$-pole which appears in $\langle O_1\rangle^0$ and $\langle O_2\rangle^0$, respectively.

After the <u>renormalization of the quark fields one has</u>

$$\langle O_1\rangle^{\text{quark field}}_{\text{ren}} = \left(1+2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right)S_1+\frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)S_1-3\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{-p^2}\right)S_2\,,$$

$$\langle O_2\rangle^{\text{quark field}}_{\text{ren}} = \left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right)S_2 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2}\right)S_2 - 3\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^2}\right)S_1\,.$$

$\rightarrow$ Still divergent after quark-field renormalization!

$\rightarrow$ the local operators O_1 and O_2 are singular objects. Additional renormalization is needed, called "renormalization of composite operators". We introduce new operatos $O^{\text{ren}}_{1,2}$:

$$O_i = \bar{Z}^{op}_{ij}O^{\text{ren}}_j\,.$$

We choose the $\bar{Z}^{op}_{ij}$ such that the matrix elements of the renormalized operators $O^{\text{ren}}_{1,2}$ become finite.

We do the operator renormalization in the $\overline{\text{MS}}$-scheme. The 2×2 matrix $\bar{Z}^{op}$ can easily be read-off:

$$\bar{Z}^{op} = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\begin{pmatrix} \frac{3}{N_c} & -3 \\ -3 & \frac{3}{N_c}\end{pmatrix}$$

As $\bar{Z}^{op}$ is a matrix, one says that O_1 and O_2 mix under renormalization: "Operator mixing".

After operator renormalization we have our finite result for $\langle O^{\text{ren}}_1\rangle$, $\langle O^{\text{ren}}_2\rangle$:

$$\begin{aligned}
\langle O^{\text{ren}}_1\rangle &= \left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right)S_1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}S_1 - 3\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}S_2\,,\\
\langle O^{\text{ren}}_2\rangle &= \left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}\right)S_2 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}S_2 - 3\frac{\alpha_s}{4\pi}\ln\frac{\mu^2}{-p^2}S_1\,,\\
S_1 &\equiv \langle O_1\rangle_{\text{tree}}\,,\quad S_2 \equiv \langle O_2\rangle_{\text{tree}}\,.
\end{aligned}$$

The renormalized amplitude in the effective theory then reads:

$$A^{\text{ren}}_{\text{eff}} = -i\langle H_{\text{eff}}\rangle = -\frac{4iG_F}{\sqrt{2}}V_{cb}V^*_{ud}\left[C_1\langle O^{\text{ren}}_1\rangle + C_2\langle O^{\text{ren}}_2\rangle\right]\,.$$

We now impose the matching condition:

$$A^{\text{ren}}_{\text{eff}} \stackrel{!}{=} A^{\text{ren}}_{\text{full}}\,.$$

The comparison of the coefficients of S_1 and S_2 leads to the Wilson coefficients C_1 and C_2:

$$C_1(\mu) = -3\frac{\alpha_s}{4\pi}\ln\frac{m_W^2}{\mu^2};\quad C_2(\mu) = 1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi}\ln\frac{m_W^2}{\mu^2}\,.$$

Remark: They do not depend on the external momentum p^2, as expected.

Notice: We only worked out the μ-dependence at $\mathcal{O}(\alpha_s)$. We did not calculate the constant (i.e. μ-independent) terms. The latter would depend on the choice of the evanescent operators.

From the explicit result

$$C_1(\mu) = -3\frac{\alpha_s}{4\pi} \ln \frac{m_W^2}{\mu^2}; \quad C_2(\mu) = 1 + \frac{3}{N_c}\frac{\alpha_s}{4\pi} \ln \frac{m_W^2}{\mu^2}$$

we see that the Wilson coefficients $C_1(\mu)$, $C_2(\mu)$ get large logarithms when choosing $\mu \approx m_b$.

One the other hand μ is an arbitrary scale. Why should we put $\mu \approx m_b$? When choosing $\mu \approx m_W$ instead, then the log's in the Wilson coefficients would be small!

We have to take into account, that the physics is not only contained in the Wilson coefficients, but also in the matrix elements of the operators!

The effective operators only contain the light fields, the heaviest one is the b-quark in our application. The matrix elements of the operators also depend logarithmically on μ:

$$\langle O_{1,2} \rangle \sim \ln \frac{\mu}{m}\,.$$

m is the mass (or external momentum) of a particle which occurs in the effective theory. m is therefore typically of $\mathcal{O}(m_b)$ and certainly not m_W (or m_t).

When choosing $\mu \approx m_b$, then the matrix elements are free of large logs. In this case, however, the Wilson coefficients contain large logs.

When choosing $\mu \approx m_W$ the situation is just vice-versa.

$\Rightarrow$ No free lunch! Large log's are simply present! These large terms have to be resummed to all orders. At $\mu \approx m_b$ these log's are in the Wilson coefficients.

In n-th order (n-gluon exchanges) the leading term of the Wilson coefficients is of the form:

$$\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^n \ln^n \frac{m_W^2}{\mu_b^2}; \qquad \mu_b = \mathcal{O}(m_b)\,.$$

These terms are called “leading logarithms (LL)”.

The so-called next-to-leading logarithms (NLL) are of the form (again n-gluon exchanges):

$$\left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^n \ln^{n-1} \frac{m_W^2}{\mu_b^2}\,.$$

What did we actually calculate so far when using this language?

$$C_1(\mu_b) = -3\frac{\alpha_s(\mu_b)}{4\pi}\ln\frac{m_W^2}{\mu_b^2}; \quad C_2(\mu_b) = 1 + \frac{3}{N_C}\frac{\alpha_s(\mu_b)}{4\pi}\ln\frac{m_W^2}{\mu_b^2}$$

So: 0 gluon contribution and 1 gluon exchange explicitly calculated. For both, $n = 0$ and $n = 1$, we are at LL precision.

In the $n = 1$ calculation we mentioned that the Wilson coefficients contain a term without a logarithm. We did not calculate it, because we didn't want to enter the business of evanescent operators. The result, e.g. for $C_2(\mu_b)$, would be of the form

$$C_2(\mu_b) = 1 + \frac{3}{N_c}\frac{\alpha_s(\mu_b)}{4\pi}\ln\frac{m_W^2}{\mu_b^2} + c\cdot\frac{\alpha_s(\mu_b)}{4\pi} \quad \text{, where } c \text{ is a number.}$$

The added term (proportional to c) is a NLL term and therefore less important.

In the following we want to consider the resummation of the LL-terms to all orders in perturbation theory.

Before really doing so, it is convenient to systematize a little bit the renormalization issues in the effective theory.

4.4 A different look at operator renormalization

We try to systematize the renormalization in the effective theory. In particular, we want to implement the counterterm formalism concerning composite operators.

As a first step all quantities in H_{eff} are understood to be bare quantities, i.e. the fields, the couplings, the masses if present. In particular, the Wilson coefficients are treated like coupling constants, i.e., as bare quantities at the starting point.

$$H_{\text{eff}} = C_i^{\text{bare}}\, O_i(q^{\text{bare}})\,; \qquad \frac{4G_F}{\sqrt{2}}V_{cb}V_{ud}^* \quad \text{omitted}$$

At this level the operators are composed of bare fields, as the notation indicates. We now write q^{bare} and C_i^{bare} in terms of the corresponding renormalized quantities

$$q^{\text{bare}} = \bar{Z}_2^{1/2}\, q\,; \quad C_i^{\text{bare}} = \bar{Z}_{ij}^c C_j\,.$$

The renormalization of the Wilson coefficients replaces the renormalization of the operators.

$$H_{\text{eff}} = \bar{Z}_2^2\, \bar{Z}_{ij}^c\, C_j O_i$$

O_i: now contains renormalized fields.

$$\boxed{H_{\text{eff}} = C_i O_i + \underbrace{(\underbrace{\bar{Z}_2^2\, Z_{ij}^c - \delta_{ij}}_{\text{starts at}\,\mathcal{O}(\alpha_s)}) C_j O_i}_{\to\,\text{counterterm}}}$$

In this formulation the renormalized matrix element $A_{\text{eff}}^{\text{ren}}$ can be compactly written as

$$A_{\text{eff}}^{\text{ren}} = -i\,\bar{Z}_2^2\,\bar{Z}_{ij}^c\,C_j \langle O_i\rangle^0\,,$$

where $\langle O_i\rangle^0$ is regularized matrix element and where $\bar{Z}_2^2\,\bar{Z}_{ij}^c$ contains the counterterms.

In the old formulation (where we were speaking about operator renormalization), we had for $A_{\text{eff}}^{\text{ren}}$ in compact notation:

$$A_{\text{eff}}^{\text{ren}} = -i\,\bar{Z}_2^2\,(\bar{Z}^{op})_{ji}^{-1}\,C_j\langle O_i\rangle^0$$

Exercise: Convince yourself that this is true!

Compare the results of the two formulations:

$$\Rightarrow \bar{Z}_{ij}^c = (\bar{Z}^{op})_{ji}^{-1} \Leftrightarrow \bar{Z}^c = (\bar{Z}^{op})^{-1\,T} \qquad \text{matrix notation}$$

$$\bar{Z}^c = 1 - \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\begin{pmatrix} \frac{3}{N_c} & -3 \\ -3 & \frac{3}{N_c} \end{pmatrix} + \mathcal{O}(\alpha_s^2)$$

Moral: Either the operators or the Wilson coefficients get renormalized, but not both!

4.5 Renormalization-group equations

The final goal is to derive a differential equation, which governs the μ-dependence of the Wilson coefficients $C_1(\mu)$ and $C_2(\mu)$: Renormalization-group equations for the Wilson coefficients. In order to solve it, we have to know how $g_s(\mu)$ depends on μ.

As in many situations the operators contain mass factors $m(\mu)$ (besides the fields of which they are composed), we need to know also the μ-dependence of the masses.

4.5.1 Running coupling $g_s(\mu)$

We are only interested in higher order QCD-effects. We start with the QCD Lagrangian, which at the beginning is expressed in terms of bare quantities.

Occuring quantities: g_s^{bare}, m_i^{bare}, ψ_i^{bare}, A_{bare}^A where $i = u, d, c, s, t, b$.

$$[g_s^{\text{bare}}] = (\text{mass})^{\frac{4-d}{2}} = (\text{mass})^{\epsilon}; \qquad (d = 4 - 2\epsilon)$$

We express the bare quantities in terms of the renormalized ones:

$$\begin{aligned} g_s^{\text{bare}} &= \mu^{\epsilon} \bar{Z}_{g_s}\, g_s \\ m_i^{\text{bare}} &= \bar{Z}_{m_i}\, m_i \\ &\text{etc.} \end{aligned}$$

We construct the counterterm Lagrangian and fix the various $(\bar{Z}-1)$-factors in such a way that the Greens-functions get finite when taking into account the counterterms.

The scale μ was not there in the original formulation; we introduced it in order to have the renormalized coupling g_s dimensionless.

Physics should be independent of the choice of μ! This only works when we accept that g_s depends on μ in a very specific way: $g_s(\mu)$

$$(1) \qquad \mu \frac{d}{d\mu} g_s^{\text{bare}} = 0 \qquad (\text{obvious})$$

Call $g_s \equiv g$ in the following (just to simplify the notation).

<u>Consequences of (1):</u>

$$\mu \frac{d}{d\mu} g^{\text{bare}} = \mu \frac{d}{d\mu}[\mu^{\epsilon} \bar{Z}_g\, g(\mu)] = 0$$

$$0 = \epsilon \mu^{\epsilon} \bar{Z}_g g(\mu) + \mu^{\epsilon}(\mu \frac{d}{d\mu} \bar{Z}_g) g(\mu) + \mu^{\epsilon} \bar{Z}_g (\mu \frac{d}{d\mu} g(\mu))$$

$$\rightarrow \mu \frac{d}{d\mu} g(\mu) = \underbrace{-\epsilon g(\mu) - \frac{1}{\bar{Z}_g}(\mu \frac{d}{d\mu} \bar{Z}_g) g(\mu)}_{=:\beta(g(\mu),\epsilon)}$$

$$\begin{aligned} \beta(g,\epsilon) &= -\epsilon g + \beta(g) \\ \beta(g) &= -g \frac{1}{\bar{Z}_g}(\mu \frac{d}{d\mu} \bar{Z}_g) \end{aligned}$$

The renormalized coupling $g(\mu)$ is finite for $\epsilon \to 0$, consequently also $\beta(g)$.

In the $\overline{\text{MS}}$ (MS) scheme, the $\bar{Z}_i$-factors depend on μ only via $g(\mu)$. They have <u>no explicit</u> μ-dependence. Furthermore, they are also independent of masses.

$$\bar{Z}_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} \bar{Z}_{i,k}(g)$$

Claim: $\beta(g) = g^2 \frac{\partial \bar{Z}_{g,1}}{\partial g}$

Note that k is always 1: in each order in g only the $1/\epsilon$-pole counts for the β-function!!

Proof:

$$\beta(g) = -g \frac{1}{\bar{Z}_g} (\mu \frac{d}{d\mu} \bar{Z}_g)$$
$$\beta(g) = -g \frac{1}{\bar{Z}_g} \underbrace{(\mu \frac{dg}{d\mu})}_{\beta(g,\epsilon)} \frac{\partial \bar{Z}_g}{\partial g}$$
$$\bar{Z}_g \beta(g) = -g\, \beta(g,\epsilon)\, \frac{\partial \bar{Z}_g}{\partial g}$$

$$\beta(g) \left[1 + \frac{\bar{Z}_{g,1}}{\epsilon} + \frac{\bar{Z}_{g,2}}{\epsilon^2} + ... \right] = -g\, \underbrace{\beta(\epsilon, g)}_{-g\epsilon + \beta(g)}\, \frac{1}{\epsilon} \left[\frac{\partial \bar{Z}_{g,1}}{\partial g} + \frac{1}{\epsilon} \frac{\partial \bar{Z}_{g,2}}{\partial g} + ... \right]$$
$$\beta(g) \left[1 + \frac{\bar{Z}_{g,1}}{\epsilon} + \frac{\bar{Z}_{g,2}}{\epsilon^2} + ... \right] = g^2 \left[\frac{\partial \bar{Z}_{g,1}}{\partial g} + \frac{1}{\epsilon} \frac{\partial \bar{Z}_{g,2}}{\partial g} + ... \right] - g\beta(g) \frac{1}{\epsilon} \left[\frac{\partial \bar{Z}_{g,1}}{\partial g} + \frac{1}{\epsilon} \frac{\partial \bar{Z}_{g,2}}{\partial g} + ... \right]$$

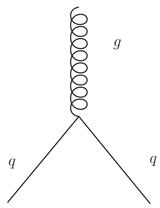
Equate the ϵ^0-terms in the above equation:

$$\rightarrow \beta(g) = g^2 \frac{\partial \bar{Z}_{g,1}}{\partial g}$$

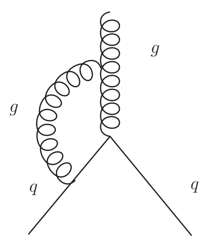
So we have:

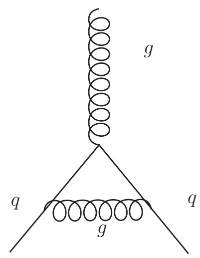
$$\mu \frac{d}{d\mu} g(\mu) = -\epsilon g + \beta(g); \quad \beta(g) = g^2 \frac{\partial \bar{Z}_{g,1}}{\partial g}; \qquad \text{with} \quad \bar{Z}_g = 1 + \frac{\bar{Z}_{g,1}}{\epsilon} + \frac{\bar{Z}_{g,2}}{\epsilon^2} + ...$$

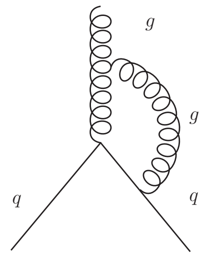
For the following, we need $\bar{Z}_g$ explicitly. To fix $\bar{Z}_g$ at order g^2, we can look e.g. at the one-loop QCD correction to the following tree-vertex:



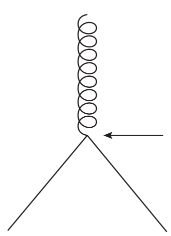
<u>1-PI vertex correction diagrams:</u>







$\bar{Z}_g$ enters the counterterm



Requirement: Diagrams + counterterms $\stackrel{!}{=}$finite

$\rightarrow$ fixes the combination $\sqrt{\bar{Z}_A \bar{Z}_2 \bar{Z}_2}\bar{Z}_g - 1$.

Need to work out $\bar{Z}_A$ (and $\bar{Z}_2$) before we can fix $\bar{Z}_g$.

But we just give the result for $\bar{Z}_g$.

<u>Result:</u>

$$\bar{Z}_{g_s} = 1 - \frac{g_s^2}{16\pi^2}\left[\frac{11}{6}N_c - \frac{2}{6}f\right]\frac{1}{\epsilon} + \mathcal{O}(g_s^4)$$

The terms proportional to g_s^2 correspond to 1-loop corrections, etc. $\bar{Z}_{g_s}$ is known up to four loops!

f: Number of flavours which run in the fermion-loop.

N_c: Number of colours ($N_c = 3$).

Exercise: Which fermion-loops are meant?!

Summary:

$$\beta(g_s) = g_s^2 \frac{\partial \bar{Z}_{g_s,1}}{\partial g_s} = g_s^2 \left(\frac{-2\, g_s}{16\pi^2} \right) \left[\frac{11}{6} N_c - \frac{2}{6} f \right] + \mathcal{O}(g_s^5)\,.$$

$\beta(g_s) = -\beta_0 \frac{g_s^3}{16\pi^2} - \beta_1 \frac{g_s^5}{(16\pi^2)^2} - \beta_2 \frac{g_s^7}{(16\pi^2)^3} - \ldots$ $\beta_0 = \frac{11 N_c - 2f}{3}; \quad \beta_1 = \frac{34}{3} N_c^2 - \frac{10}{3} N_c f - 2 C_F f; \quad C_F = \frac{4}{3}$
$\underline{d = 4}$ $\mu \frac{d}{d\mu} g_s(\mu) = \beta(g_s) = -\beta_0 \frac{g_s^3}{16\pi^2} - \beta_1 \frac{g_s^5}{(16\pi^2)^2} - \ldots$
Equation for $\alpha_s(\mu)$ $\quad \alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}$; $\mu \frac{d}{d\mu} \alpha_s(\mu) = -2\beta_0 \frac{\alpha_s^2}{4\pi} - 2\beta_1 \frac{\alpha_s^3}{(4\pi)^2} - \ldots$

The last line is called "renormalization group equation (RGE) for $\alpha_s(\mu)$".

Solution of the RGE for $\alpha_s(\mu)$

We only keep the leading term of the β-function.

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = -2\beta_0 \frac{\alpha_s^2(\mu)}{4\pi}$$

$\rightarrow$ Ordinary differential equation of first order. Need one initial condition: $\alpha_s(\mu_0)$

Put: $\mu = \mu_0\, e^t; \qquad t = \ln \frac{\mu}{\mu_0}$

$$\mu \frac{d}{d\mu} = \mu \underbrace{\frac{dt}{d\mu}}_{\frac{1}{\mu}} \frac{d}{dt} = \frac{d}{dt}$$

$$\frac{d}{dt} \alpha_s(t) = -2\beta_0 \frac{\alpha_s^2(t)}{4\pi}; \quad \alpha_s(t=0) \hat{=} \alpha_s(\mu_0)$$

$$\frac{\dot{\alpha}_s(t)}{\alpha_s^2(t)} = -\frac{2\beta_0}{4\pi}$$

$$\int_0^t \frac{\dot{\alpha}_s(t')}{\alpha_s^2(t')} dt' = -\frac{2\beta_0}{4\pi} t$$

$$\int_{\alpha_s(0)}^{\alpha_s(t)} \frac{1}{\alpha_s^2} d\alpha_s = -\frac{2\beta_0}{4\pi} t$$

$$
\begin{aligned}
-\frac{1}{\alpha_s}\bigg|_{\alpha_s(0)}^{\alpha_s(t)} &= -\frac{2\beta_0}{4\pi}t \\
-\frac{1}{\alpha_s(t)} + \frac{1}{\alpha_s(0)} &= -\frac{2\beta_0}{4\pi}t \\
\frac{1}{\alpha_s(t)} &= \frac{1}{\alpha_s(0)} + \frac{\beta_0}{2\pi}t \\
\alpha_s(t) &= \frac{\alpha_s(0)}{1 + \frac{\beta_0}{2\pi}\alpha_s(0)t}
\end{aligned}
$$

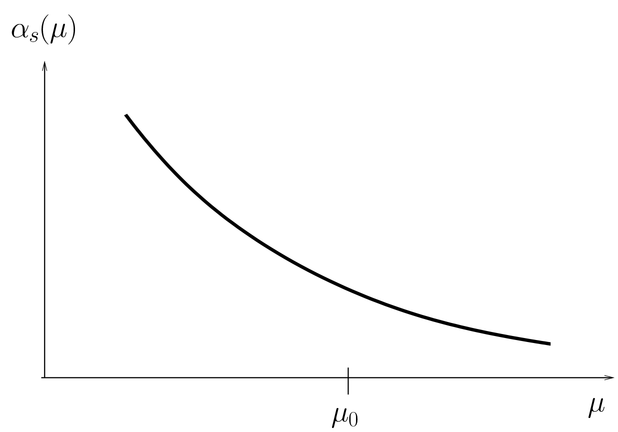
or

$$
\boxed{\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_0)\ln\frac{\mu}{\mu_0}}} \tag{4.1}
$$

In high energy processes it is favorable to choose $\mu = E$, where E is the characteristic energy scale of the process. When choosing a μ-value, which has nothing to do with the relevant energies of the process, the predictions contain explicit $\ln\frac{E}{\mu}$ terms which are large.

$\rightarrow$ A convenient choice of μ absorbs these logarithms into $\alpha_s(\mu) \rightarrow$ better behaved perturbation theory.

Energy larger $\rightarrow \mu$ larger; $\beta_0 = \frac{11N_c - 2f}{3} > 0$.



From the LEP precision measurements done at the Z-pole, $\alpha_s(m_Z)$ was extracted:

$$
\alpha_s(m_Z) = 0.118 \pm 0.003 \qquad (\overline{\text{MS}}, f = 5).
$$

We will use this value as initial condition of the RGE, i.e., $\mu_0 = m_Z$.

Expand eq. (4.1):

$$
\alpha_s(\mu) = \alpha_s(\mu_0)\left[1 - \frac{\beta_0}{2\pi}\alpha_s(\mu_0)\ln\frac{\mu}{\mu_0} + \left(\frac{\beta_0}{2\pi}\right)^2 \alpha_s^2(\mu_0)\ln^2\frac{\mu}{\mu_0} + ...\right]
$$

$\rightarrow$ leading logarithms (LL) resummed!!

If we had taken into account the next term in the β-function, i.e. the one involving β_1, then the RGE would also resum the next-to-leading logarithms (NLL).

4.5.2 Running masses

The bare masses m_i^{bare} and the renormalized masses m_i are related according to

$$m_i^{\text{bare}} = \bar{Z}_{m_i} m_i \qquad \text{(we skip the label } i\text{)}$$

$\bar{Z}_m$ has an expansion in powers of g (and in inverse powers of ϵ), like $\bar{Z}_g$. Again the bare mass m^{bare} is independent of μ. This only works if the renormalized mass m is μ-dependent: $m(\mu)$.

$$\begin{aligned}
0 &= \mu\frac{d}{d\mu}m^{\text{bare}} = \mu\frac{d}{d\mu}[\bar{Z}_m m(\mu)] \\
0 &= (\mu\frac{d}{d\mu}\bar{Z}_m)m + \bar{Z}_m(\mu\frac{d}{d\mu}m) \\
\mu\frac{d}{d\mu}m(\mu) &= -\underbrace{\frac{1}{\bar{Z}_m}(\mu\frac{d}{d\mu}\bar{Z}_m)}_{=:\gamma_m} m(\mu)
\end{aligned}$$

$$\underline{\mu\frac{d}{d\mu}m(\mu) = -\gamma_m\, m(\mu);} \qquad \gamma_m = \frac{1}{\bar{Z}_m}(\mu\frac{d}{d\mu}\bar{Z}_m).$$

Like $\bar{Z}_g$, also $\bar{Z}_m$ depends in the $\overline{\text{MS}}$ (MS)-scheme on μ only via g. Similar to the derivation of $\beta(g)$, one can show that

$$\boxed{\gamma_m(g) = -g\frac{\partial \bar{Z}_{m,1}}{\partial g}}$$

$\left(\bar{Z}_m = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k}\bar{Z}_{m,k}\right)$

<u>Exercise:</u> Show it!

The renormalization factor $\bar{Z}_m$ to order g_s^2 reads:

$$\begin{aligned}
\bar{Z}_m &= 1 - \frac{g_s^2}{16\pi^2}\,C_F\,\frac{3}{\epsilon} + \mathcal{O}(g_s^4); \qquad \left(C_F = \frac{4}{3}.\right) \\
\Rightarrow \gamma_m &= -g_s\left(-\frac{2g_s}{16\pi^2}C_F\,3\right) + \mathcal{O}(g_s^4)
\end{aligned}$$

$$\gamma_m = \frac{\alpha_s}{4\pi} 6\, C_F + \mathcal{O}(\alpha_s^2)\,.$$

Generally, γ_m has an expansion is α_s:

$$\gamma_m = \frac{\alpha_s}{4\pi}\gamma_m^{(0)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \gamma_m^{(1)} + ...$$
$$\gamma^{(0)} = 6\, C_F\,; \qquad \gamma^{(1)} = C_F \left(3\, C_F + \frac{97}{3}N_c - \frac{10}{3}f\right)\,.$$

To summarize: The renormalization group equation for the running mass $m(\mu)$ reads

$$\boxed{\mu \frac{d}{d\mu} m(\mu) = -\gamma_m(g(\mu))\, m(\mu)}$$

<u>Solution:</u> At $\mu = \mu_0$ we impose the initial condition $m(\mu_0)$.

$$m(\mu) = m(\mu_0) \exp\left[-\int_{g(\mu_0)}^{g(\mu)} dg' \frac{\gamma_m(g')}{\beta(g')}\right] \qquad \text{(exact expression!)}$$

<u>Check:</u>

$$\mu \frac{d}{d\mu} m(\mu) = \underbrace{m(\mu_0) e^{-\int \cdots}}_{m(\mu)} \frac{[-\gamma_m(g(\mu))]}{\beta(g(\mu))} \underbrace{\mu \frac{d}{d\mu} g(\mu)}_{\beta(g(\mu))}$$
$$\mu \frac{d}{d\mu} m(\mu) = -\gamma_m(g(\mu))\, m(\mu) \qquad \text{(o.k.!)}$$

Solution when using lowest order expressions for the β- and the γ_m-functions:

$$m(\mu) = m(\mu_0) \left[\frac{\alpha_s(\mu))}{\alpha_s(\mu_0)}\right]^{\frac{\gamma_m^{(0)}}{2\beta_0}}\,.$$

<u>Exercise:</u> Show this!

If one expands this formula, one sees that also here the leading logarithms get resummed.

$\rightarrow$ Next: RGE for $C_1(\mu)$ and $C_2(\mu)$.

4.5.3 Renormalization group equation for $C_1(\mu), C_2(\mu)$

Starting point:

$$C_i^{\text{bare}} = \bar{Z}^c_{ij} C_j$$

$$\vec{C}^{\text{bare}} = \bar{Z}^c \vec{C} \quad \text{(matrix notation)}$$

$$0 = \mu \frac{d}{d\mu} \vec{C}^{\text{bare}} = \left(\mu \frac{d}{d\mu} \bar{Z}^c\right) \vec{C} + \bar{Z}^c \left(\mu \frac{d}{d\mu} \vec{C}\right)$$

$$\underline{\mu \frac{d}{d\mu} \vec{C} = -(\bar{Z}^c)^{-1} \left(\mu \frac{d}{d\mu} \bar{Z}^c\right) \vec{C}}$$

Definition:

$$\hat{\gamma} := (\bar{Z}^{op})^{-1} \left(\mu \frac{d}{d\mu} \bar{Z}^{op}\right) \qquad \text{anomalous dimension matrix}$$

$$\bar{Z}^{op} = [(\bar{Z}^c)^{-1}]^T$$

$$\to \hat{\gamma} = (\bar{Z}^c)^T \mu \frac{d}{d\mu} (\bar{Z}^c)^{-1T}$$

$$\hat{\gamma}^T = \left(\mu \frac{d}{d\mu} (\bar{Z}^c)^{-1}\right) \bar{Z}^c = -(\bar{Z}^c)^{-1} \left(\mu \frac{d}{d\mu} \bar{Z}^c\right)$$

$$\to \boxed{\mu \frac{d\vec{C}}{d\mu} = \hat{\gamma}^T \vec{C}}$$

Solution: $\vec{C}(\mu) = \hat{U}(\mu, \mu_0)\, \vec{C}(\mu_0)$

$$\hat{U}(\mu, \mu_0) = T_{g'} \exp\left[\int_{g(\mu_0)}^{g(\mu)} dg' \, \frac{\hat{\gamma}^T(g')}{\beta(g')}\right]; \qquad \text{evolution matrix}$$

$\hat{\gamma}$ from $\bar{Z}^{op}$:

$$\bar{Z}^{op} = \mathbb{1} + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} \bar{Z}^{op}_k(g)\,; \quad \mu\text{-dependent only via } g(\mu) \text{ in } \overline{\text{MS}} \text{ (MS) scheme.}$$

Exercise: $\hat{\gamma}(g) = -g \frac{\partial \bar{Z}^{op}_1}{\partial g}$

$$\bar{Z}^{op} = \mathbb{1} + \frac{g^2}{16\pi^2} \frac{1}{\epsilon} \begin{pmatrix} \frac{3}{N_c} & -3 \\ -3 & \frac{3}{N_c} \end{pmatrix} + \mathcal{O}(g^4)\,.$$

$$\hat{\gamma} = -\frac{2g^2}{16\pi^2} \begin{pmatrix} \frac{3}{N_c} & -3 \\ -3 & \frac{3}{N_c} \end{pmatrix} + \mathcal{O}(g^4)$$

$$\hat{\gamma} = \frac{\alpha_s}{4\pi}\begin{pmatrix} -\frac{6}{N_c} & 6 \\ 6 & -\frac{6}{N_c} \end{pmatrix} + \mathcal{O}(\alpha_s^2)$$

$$\underline{\hat{\gamma} = \frac{\alpha_s}{4\pi}\hat{\gamma}^0 + \mathcal{O}(\alpha_s^2); \quad \hat{\gamma}^0 = \begin{pmatrix} -\frac{6}{N_c} & 6 \\ 6 & -\frac{6}{N_c} \end{pmatrix}}$$

$$\hat{U}(\mu,\mu_0) = T_{g'} \exp\left[\int_{g(\mu_0)}^{g(\mu)} dg' \frac{[\frac{g'^2}{16\pi^2}\hat{\gamma}^{0T} + \mathcal{O}(g'^4)]}{[-\beta_0 \frac{g'^3}{16\pi^2} + \mathcal{O}(g'^5)]}\right]$$

$$\hat{U}(\mu,\mu_0) = T_{g'} \exp\left[\int_{g(\mu_0)}^{g(\mu)} dg' \frac{-\hat{\gamma}^{0T}}{\beta_0\, g'}[1 + \mathcal{O}(g'^2)]\right]$$

<u>at lowest order:</u>

$$\hat{U}(\mu,\mu_0) = \exp\left[\frac{-\hat{\gamma}^{0T}}{\beta_0}\int_{g(\mu_0)}^{g(\mu)} \frac{dg'}{g'}\right] = \exp\left[\frac{-\hat{\gamma}^{0T}}{\beta_0}\ln\frac{g(\mu)}{g(\mu_0)}\right]$$

$$\underline{\hat{U}(\mu,\mu_0) = \exp\left[\frac{-\hat{\gamma}^{0T}}{2\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]}$$

Diagonalize $\hat{\gamma}^{0T}$, i.e. find a matrix V such that

$$\hat{\gamma}^{0T} = V\,\hat{\gamma}_D^0\,V^{-1}\,; \qquad \text{with } \hat{\gamma}_D^0 \text{ diagonal.}$$

$$\to \hat{U}(\mu,\mu_0) = V\left(\exp\left[\frac{-\hat{\gamma}_D^0}{2\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]\right)V^{-1}$$

$$\underline{\hat{U}(\mu,\mu_0) = V\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}^{\frac{\vec{\gamma}^{(0)}}{2\beta_0}}\right)_D V^{-1}}$$

$\vec{\gamma}^{(0)} \equiv \mathrm{diag}(\hat{\gamma}_D^0)$. Explicitly V and V^{-1} read

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\,; \quad V^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\,.$$

Using this, we get

$$\hat{\gamma}_D^0 = \begin{pmatrix} \frac{6(N_c-1)}{N_c} & 0 \\ 0 & -\frac{6(N_c+1)}{N_c} \end{pmatrix} \to \vec{\gamma}^{(0)} = \left(\frac{6(N_c-1)}{N_c}, -\frac{6(N_c+1)}{N_c}\right)$$

$$\beta_0 = \frac{11N_c - 2f}{3}; \qquad f = 5 \quad (u, d, c, s, b, \cancel{t}).$$

Note that the top quark t is integrated out.

$$\begin{aligned}
\Rightarrow \beta_0 &= \frac{23}{3} \\
\vec{\gamma}^{(0)} &= (4, -8) \\
\hat{U}(\mu, \mu_0) &= V \begin{pmatrix} r^{\frac{6}{23}} & 0 \\ 0 & r^{\frac{-12}{23}} \end{pmatrix} V^{-1}; \qquad r := \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \\
\underline{\hat{U}(\mu, \mu_0) = \frac{1}{2} \begin{pmatrix} r^{\frac{6}{23}} + r^{\frac{-12}{23}} & r^{\frac{6}{23}} - r^{\frac{-12}{23}} \\ r^{\frac{6}{23}} - r^{\frac{-12}{23}} & r^{\frac{6}{23}} + r^{\frac{-12}{23}} \end{pmatrix}}.
\end{aligned}$$

$\underline{\mu = \mu_W \, (= \mathcal{O}(m_W))}$:

- The Wilson coefficients do not contain large logarithms.
- The matrix elements of the operators have large logarithms.

$\underline{\mu = \mu_b \, (= \mathcal{O}(m_b)) :}$

- The Wilson coefficients contain large logarithms.
- The matrix elements of the operators are free of large logarithms.

$\Rightarrow$ Do matching at $\mu = \mu_W : C_i(\mu_W) \xrightarrow{\text{RGE}}$ Wilson coefficients at $\mu = \mu_b : C_i(\mu_b)$; i.e.

$$\vec{C}(\mu_b) = \hat{U}(\mu_b, \mu_W)\, \vec{C}(\mu_W)$$

$\underline{\hat{U}(\mu_b, \mu_W)}$: contains $\left(\alpha_s(\mu_b) \ln \frac{\mu_W^2}{\mu_b^2}\right)^N$ in resummed form.

$\underline{\vec{C}(\mu_W)}$: α_s-corrections “small”, no large logarithms $\rightarrow$ At LL-precision, it is not even necessary to calculate the α_s-corrections at $\mu = \mu_W$!

Therefore: $C_1(\mu_W) = 0$ and $C_2(\mu_W) = 1$.

At the scale $\mu = \mu_b$ we then have

$$C_1(\mu_b) = \frac{1}{2}\left[\eta^{6/23} - \eta^{-12/23}\right]; \qquad C_2(\mu_b) = \frac{1}{2}\left[\eta^{6/23} + \eta^{-12/23}\right],$$

with $\eta = \alpha_s(\mu_W)/\alpha_s(\mu_b)$; The running $\alpha_s(\mu)$ reads

$$\alpha_s(\mu) = \frac{\alpha_s(m_Z)}{1 - \beta_0 \frac{\alpha_s(m_Z)}{2\pi} \ln \frac{m_Z}{\mu}}; \quad \alpha_s(m_Z) = 0.118 \pm 0.003,\ m_Z = (91.1867 \pm 0.0021) \text{ GeV, (LEP-I)}.$$

Exercise: We previously calculated $C_i(\mu)$ explicitly in order α_s precision; in particular we worked out the $\alpha_s \ln \mu$-term. Expand the resummed result and show that the $\alpha_s \ln \mu$ term indeed coincides.

Example: Choose the matching scale $\mu_W = m_W = 80.4$ GeV, $\mu_b = 5$ GeV.

$$C_1(m_W) = 0; \quad C_2(m_W) = 1$$
$$C_1(5\text{ GeV}) = -0.221; \quad C_2(5\text{ GeV}) = 1.093$$
$$[\alpha_s(m_Z) = 0.118; \quad \alpha_s(m_W) = 0.120; \quad \alpha_s(5\text{ GeV}) = 0.203.]$$

4.6 Summary

H_{eff} for $b \to c\bar{u}d$ (or $bu \to cd$)

$$H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \left[C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)\right] + \xcancel{\text{dim. 8,10,.. operators}}$$

$\mu = \mu_W$: Do the matching, i.e. fix $C_i(\mu_W)$, where μ_W is of the order of m_W.

$\big\downarrow$ Do RGE-evolution; $\hat{U}(\mu_b, \mu_W)$ is at work in this step

$\mu = \mu_b$: $C_i(\mu_b)$ are obtained, the large logarithms are contained in $C_i(\mu_b)$ in resummed form.

It remains to calculate the matrix elements of the operators: $\langle O_i(\mu_b)\rangle$:

This is the most difficult problem, because non-pertubative in general!

- For inclusive $\bar{B}^0$-decays: $\bar{B}^0 \to X_c$

  X_c : hadronic final state with charmness=1

As $m_b \gg \Lambda_{QCD}$ ($\Lambda_{QCD} \sim$ binding energy in the $\bar{B}^0$-meson): The hadronic matrix element is dominated by the quark transition $b \to c\bar{u}d$:

$$\Gamma(\bar{B}^0 \to X_c) = \underbrace{\Gamma(b \to c\bar{u}d)}_{\text{dominant contr.}} + \underbrace{\mathcal{O}\left(\frac{\Lambda^2_{QCD}}{m_b^2}\right)}_{\substack{\text{worked out:} \\ <4\% \text{ correction}}} .$$

- For exclusive decays: $\bar{B}^0 \to D^+\pi^-$

  $\langle D^+\pi^- | O_i(\mu) | \bar{B}^0 \rangle$ not calculable from first principles! Lattice not ready yet.

  Factorization Ansätze, other models.

Usually, for inclusive decays the perturbative corrections are more important than the power corrections of the form Λ^2_{QCD}/m_b^2. We now state what is needed to calculte the inclusive decay width in leading-logarithmic (LL) and next-to-leading logarithmic (NLL) approximation.

LL approximation:

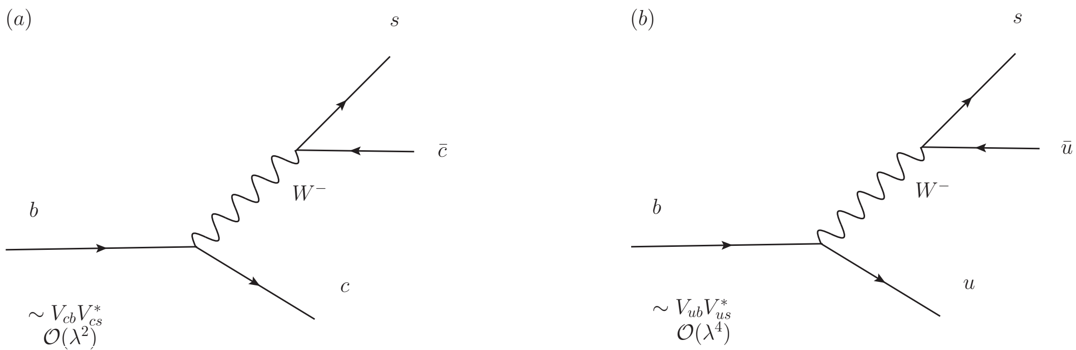
- Matching to order α_s^0 precision.
- RGE with $\hat{\gamma} = \frac{\alpha_s}{4\pi}\hat{\gamma}^{(0)} + \cancel{(\frac{\alpha_s}{4\pi})^2\hat{\gamma}^{(1)}} + \cancel{(\frac{\alpha_s}{4\pi})^3\hat{\gamma}^{(2)}} + ...$
- Evaluation of the matrix elements at the scale μ_b to order α_s^0 precision.

NLL approximation:

Each of the three steps needs to be improved by an additional power of α_s, i.e.,

- Matching to order α_s^1 precision.
- RGE with $\hat{\gamma} = \frac{\alpha_s}{4\pi}\hat{\gamma}^{(0)} + (\frac{\alpha_s}{4\pi})^2\hat{\gamma}^{(1)} + \cancel{(\frac{\alpha_s}{4\pi})^3\hat{\gamma}^{(2)}} + ...$
- Evaluation of the matrix elements at the scale μ_b to order α_s^1 precision.

5 Effective Hamiltonian for $\Delta B = \Delta S = 1$ transitions



λ : Wolfenstein-parameter $\lambda \approx 0.2$; neglect the u-contribution.

$$V_{cb}V_{cs}^* + V_{tb}V_{ts}^* + V_{ub}V_{us}^* = 0 \qquad \text{(exact due to the unitarity of CKM matrix)}$$
$$\to V_{cb}V_{cs}^* \cong -V_{tb}V_{ts}^* \,.$$

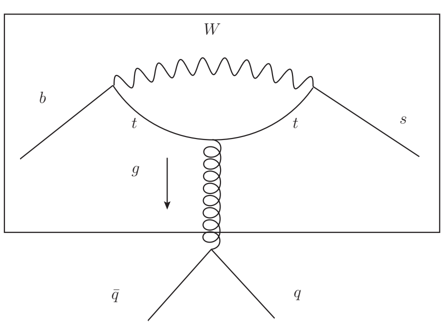
In analogy to the case $b \to c\bar{u}d$, diagram (a) gives rise to the following operators:

$$H_{eff}^{CC} = \frac{4G_F}{\sqrt{2}} \underbrace{V_{cb}V_{cs}^*}_{-V_{tb}V_{ts}^*} [C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)]$$
$$O_1 = \bar{s}_\alpha \gamma_\mu L c_\beta \, \bar{c}_\beta \gamma^\mu L \, b_\alpha$$
$$O_2 = \bar{s}_\alpha \gamma_\mu L c_\alpha \, \bar{c}_\beta \gamma^\mu L \, b_\beta$$

The Wilson coefficients $C_1(\mu), C_2(\mu)$ are the same as those for $b \to c\bar{u}d$.

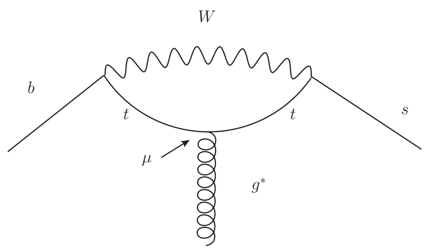
O_1 and O_2 are called current-current (CC) operators.

Note that there additional operators (for $\Delta B = \Delta S = 1$) generated by QCD:



Peguin-diagram

There are two form-factors in the limit $m_s \to 0$:



$$\sim \left[\bar{s}(q^2\gamma_\mu - q_\mu \not{q})L\,T^A b)\right] \cdot F_1^g(m_t, m_W) + \left[\bar{s}(i\sigma_{\mu\nu}q^\nu m_b R)\,T^A b\right] \cdot F_2^g(m_t, m_W) \tag{5.1}$$

The first structure leads to a local operator when attaching "the feet" of the penguin:

$$\bar{s}(q^2\gamma_\mu - q_\mu \not{q})LT^A b \quad \underbrace{\frac{1}{q^2}}_{\substack{\text{from}\\ \text{gluon-prop.}}} \quad \bar{q}\gamma^\mu T^A q$$

$$= (\bar{s}\gamma_\mu \cancel{q^2} LT^A b)\cancel{\frac{1}{q^2}}(\bar{q}\gamma^\mu T^A q)$$

$$= (\bar{s}\gamma_\mu LT^A b)(\bar{q}\gamma^\mu \mathbb{1} T^A q)\,; \qquad q \in \{u, d, c, s, b\}$$

<u>Exercise:</u> Why did we "ignore" the contribution from $q_\mu \not{q}$ in the equation above?

<u>Lorentz structure:</u> Decompose the result w.r.t. L and R: $\mathbb{1} = L + R$.

<u>Colour structure:</u> Decompose the color structure by using $T^A_{\alpha\beta}T^A_{\gamma\delta} = -\frac{1}{2N_c}\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}$.

By doing these decompositions, four operators are generated, the so-called QCD-penguin operators O_3, O_4, O_5 and O_6:

$$O_3 = (\bar{s}_\alpha\gamma^\mu L b_\alpha)\sum_q(\bar{q}_\beta\gamma_\mu L q_\beta)\,,$$

$$O_4 = (\bar{s}_\alpha\gamma^\mu L b_\beta)\sum_q(\bar{q}_\beta\gamma_\mu L q_\alpha)\,,$$

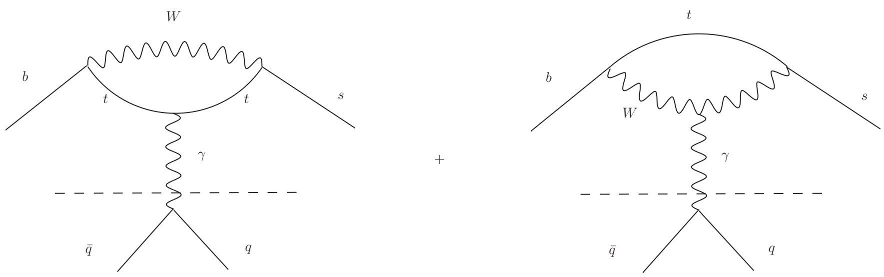
$$O_5 = (s_\alpha\gamma^\mu L b_\alpha)\sum_q(\bar{q}_\beta\gamma_\mu R q_\beta)\,,$$

$$O_6 = (s_\alpha\gamma^\mu L b_\beta)\sum_q(q_\beta\gamma_\mu R q_\alpha)\,.$$

The structure proportional to F_2^g in (5.1) leads (in position space) to the operator ($m_s = 0$)

$$O_8 = \frac{g_s}{16\pi^2}m_b\,\bar{s}_\alpha\sigma^{\mu\nu}R\,T^A_{\alpha\beta}b_\beta\,G^A_{\mu\nu}\,.$$

QCD has generated the operators $O_3 - O_6$ and O_8. In a similar way, also the electro-weak interaction generates new operators:



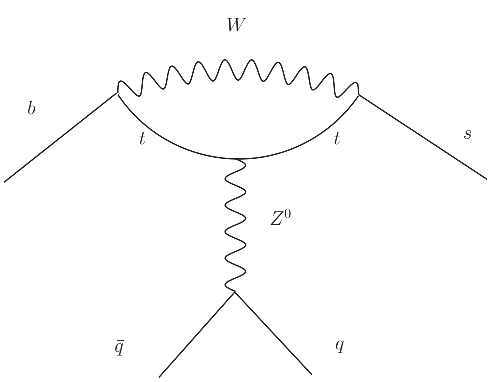
$$\sim \left[\bar{s}(q^2\gamma_\mu - q_\mu \not{q})Lb\right] F_1^\gamma(m_t, m_W)$$
$$+ \left[\bar{s}(i\sigma_{\mu\nu}q^\nu m_b R)b\right] F_2^\gamma(m_t, m_W)$$

The structure proportional to F_2^γ leads to the operator

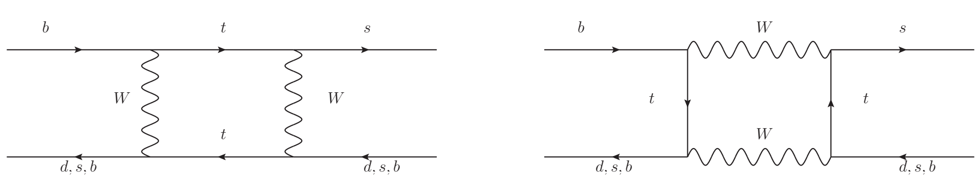
$$O_7 = \frac{e}{16\pi^2} m_b\, \bar{s}_\alpha \sigma^{\mu\nu} R\, b_\alpha\, F_{\mu\nu}\,.$$

The first structure ($\sim F_1^\gamma$) leads again to 4-Fermi-operators, when attaching the feet of the penguin.

When exchanging a Z^0 instead of a γ, also a local 4-Fermion operator is generated which is of the same order: $G_F\alpha$



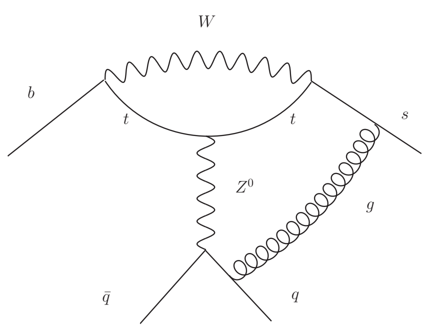
In this order, there are also W-boxes, which lead to 4-Fermi-operators:



The four-Fermi operators, which are generated through $\gamma, Z^0, W^\pm$ exchange, are called electro-weak penguin operators. They read

$$\begin{aligned}
O_9 &= \tfrac{3}{2}(\bar{s}_\alpha \gamma_\mu L b_\alpha) \textstyle\sum_q e_q (\bar{q}_\beta \gamma^\mu R\, q_\beta) \\
O_{10} &= \tfrac{3}{2}(\bar{s}_\alpha \gamma_\mu L b_\beta) \textstyle\sum_q e_q (\bar{q}_\beta \gamma^\mu R\, q_\alpha) \\
O_{11} &= \tfrac{3}{2}(\bar{s}_\alpha \gamma_\mu L b_\alpha) \textstyle\sum_q e_q (\bar{q}_\beta \gamma^\mu L\, q_\beta) \\
O_{12} &= \tfrac{3}{2}(\bar{s}_\alpha \gamma_\mu L b_\beta) \textstyle\sum_q e_q (\bar{q}_\beta \gamma^\mu L\, q_\alpha)
\end{aligned}$$

Remark: Strictly speaking, the electro-weak interaction only generates O_9 and O_{11}. O_{10} and O_{12} only come into the game when switching on QCD[1]. An example of a diagram which generates O_{10} and O_{12} is the following:



generates O_{10}, O_{12}.

Remark: Instead of a quark-pair ($\bar{q}q$), one also could have a lepton-pair, as for example in the process $b \to s\mu^+\mu^-$.

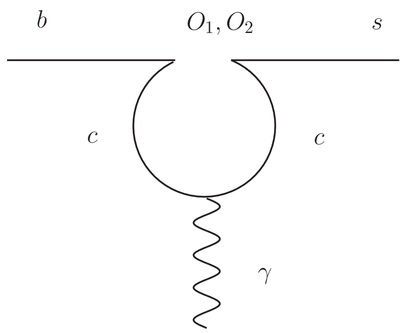
In this case operators analogous to O_9 and O_{11} are generated, while operators analogous to O_{10} and O_{12} are not there. Why not?

[1] This is similar as in section 4.1: In absence of QCD one only had O_2; O_1 enters the game when switching on QCD.

5.1 Process $b \to s\gamma$

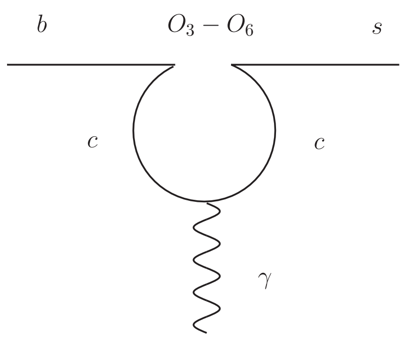
Which of the operators just discussed should be included in this case? Consider the matrix elements of the various operators O_i: $\langle s\gamma|O_i|b\rangle$

<u>O_1, O_2 :</u>



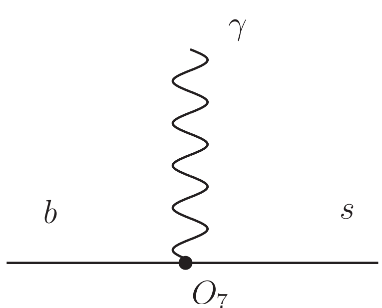
$\sim \underbrace{G_F e}_{\text{strength w.r.t electro-weak}}$

<u>$O_3 - O_6$:</u>



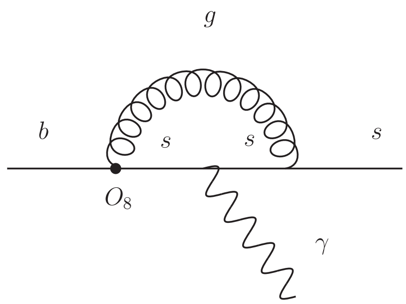
$\sim G_F e$

<u>O_7 :</u>



$\sim G_F e$

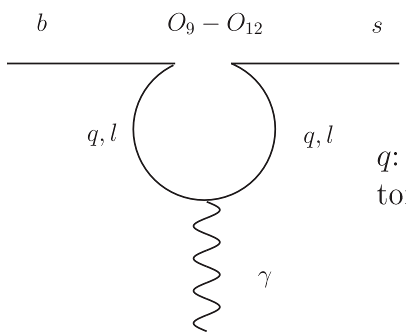
$\underline{O_8 :}$



$\sim G_F e$

Note that the factor G_F comes from the prefactor in the effective Hamiltonian, not from the operators.

$\underline{O_9 - O_{12} :}$



$\sim G_F \cdot e^2 e$

q: quarks, l: charged leptons

The contribution to the amplitude is suppressed by $\alpha = \frac{1}{137}$.

As the operators $O_9 - O_{12}$ give suppressed (in α) contributions to $b \to s\gamma$, one only takes into account $O_1 - O_8$:

$$H_{\text{eff}}(b \to s\gamma) = -\frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_{i=1}^{8} C_i(\mu) O_i(\mu); \quad \mu : \text{Renormalization-scale (separation scale)}$$

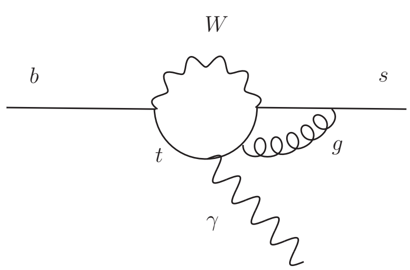
$$O_1 = \bar{s}_\alpha \gamma_\mu L\, c_\beta \bar{c}_\beta \gamma^\mu L\, b_\alpha$$
$$O_2 = \bar{s}_\alpha \gamma_\mu L\, c_\alpha \bar{c}_\beta \gamma^\mu L\, b_\beta$$
$$O_3 = \bar{s}_\alpha \gamma_\mu L\, b_\alpha \sum_q \bar{q}_\beta \gamma^\mu L\, q_\beta$$
$$O_4 = \bar{s}_\alpha \gamma_\mu L\, b_\beta \sum_q \bar{q}_\beta \gamma^\mu L\, q_\alpha$$
$$O_5 = \bar{s}_\alpha \gamma_\mu L\, b_\alpha \sum_q \bar{q}_\beta \gamma^\mu R\, q_\beta$$

$$O_6 = \bar{s}_\alpha \gamma_\mu L\, b_\beta \sum_q \bar{q}_\beta \gamma^\mu R\, q_\alpha$$

$$O_7 = \frac{e}{16\pi^2} m_b(\mu) \bar{s}_\alpha \sigma^{\mu\nu} R b_\alpha F_{\mu\nu} \qquad F_{\mu\nu} : \text{photonic field-strength}$$

$$O_8 = \frac{g_s(\mu)}{16\pi^2} m_b(\mu) \bar{s}_\alpha \sigma^{\mu\nu} R T^A_{\alpha\beta} b_\beta G^A_{\mu\nu} \qquad G^A_{\mu\nu} : \text{gluonic field-strength}$$

$\underline{b \to s\gamma}$ (full theory)



$m_b \ll m_t, m_W; M = m_t$ or m_W

large logarithms: $\ln \frac{m_b}{M}$

The large logarithms have to be resummed.

Proceed as in section 4.1, i.e.,

$\underline{C_i(\mu_W)}$ at the matching scale $\mu = \mu_W$ $(\mathcal{O}(m_W, m_t))$

$\Big\downarrow$ RGE

$\underline{C_i(\mu_b)}$ at the lower scale $\mu = \mu_b$ $(\mathcal{O}(m_b))$

The matrix elements $\langle O_i(\mu_b)\rangle$ are calculated at the low scale μ_b.

$C_i(\mu_b)$ contains the large logarithms in resummed form.

In leading-logarithmic precision: $[(\alpha_s \ln \frac{\mu_W}{\mu_b})^N]$

- Matching: $C_i(\mu_W)$ in $\mathcal{O}(\alpha_s^0)$.
- Anomalous dimension matrix: $\hat{\gamma} = \underbrace{\frac{\alpha_s}{4\pi}\hat{\gamma}^0}_{\text{sufficient}}$ $(8 \times 8$ matrix$)$.

- Matrix elements: $\langle s\gamma|O_i(\mu_b)|b\rangle)$ in $\mathcal{O}(\alpha_s^0)$.

5.1.1 Matching at $\mathcal{O}(\alpha_s^0)$

C_1 is only induced at order α_s; the same is true for $C_3 - C_6$.

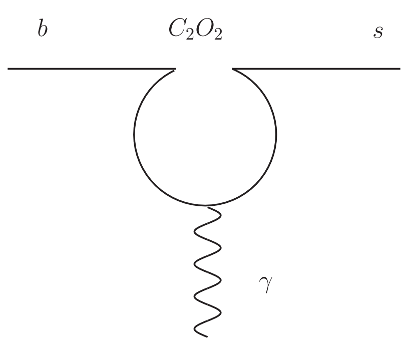
$$
\begin{aligned}
&C_1(\mu_W) = C_3(\mu_W) = C_4(\mu_W) = C_5(\mu_W) = C_6(\mu_W) = 0\,,\\
&C_2(\mu_W) = 1\,.
\end{aligned}
$$

We now want to determine $C_7(\mu_W)$.

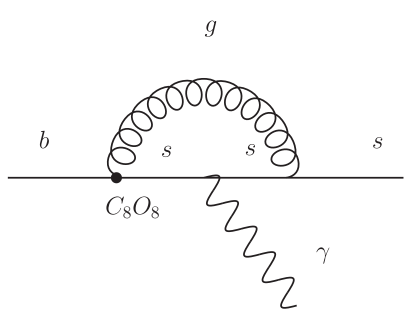
To do so, we calculate $b \to s\gamma$ at order α_s^0 in the full theory and also in the effective theory, at scale $\mu = \mu_W$.

Effective theory:

At this scale only the coefficients of O_2, O_7, O_8 are different from zero.

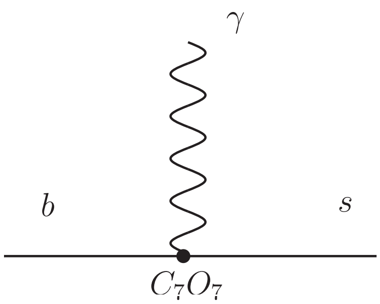


The diagram is identically zero for an on-shell photon (in d-dimensions).

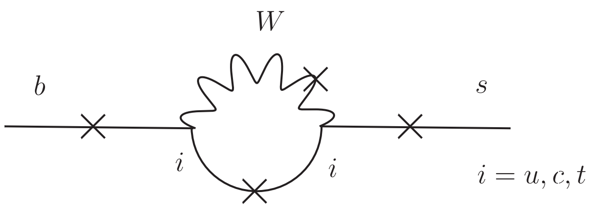


is of order α_s^1 : too high!

Therfore: Only O_7 contributes in the effective theory at order α_s^0. The corresponding diagram is:



<u>Full theory:</u>



$\times$: possible place for γ-emission $\rightarrow$ 4 diagrams

The result for the amplitude A is of the form

$$A = V_{tb} V_{ts}^* f(m_t; m_W) + V_{cb} V_{cs}^* f(m_c; m_W) + V_{ub} V_{us}^* f(m_u; m_W)\,.$$

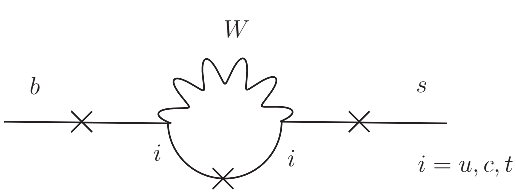
A would be zero for $m_u = m_c = m_t$!

The $1/\epsilon$-pole in $f(m_t, m_W)$ is independent of m_i. Due to the unitarity relation $V_{tb} V_{ts}^* + V_{cb} V_{cs}^* + V_{ub} V_{us}^* = 0$ the result is finite. The function f has a well-defined limit when $m_i \rightarrow 0$. To a very good approximation one can put $m_u = m_c = 0$. Doing the calculation in the full theory and comparing it with the effective theory one finds:

$$C_7(\mu_W) = \frac{z[6z(3z-2)\ln z - (z-1)(8z^2+5z-7)]}{24(z-1)^4}$$
$$z = \frac{m_t^2}{m_W^2}$$

Although not really needed for $b \rightarrow s\gamma$, what would be $C_8(\mu_W)$?

<u>$C_8(\mu_W) =$?</u> The calculation is very similar. Just consider $b \rightarrow sg$ instead of $b \rightarrow s\gamma$.



$\times$: possible location for g-emission $\rightarrow$ 3 diagrams

$$\Big\| \; C_8(\mu_W) = -\tfrac{z[6z\ln z+(z-1)(z^2-5z-2)]}{8(z-1)^4}$$

(In the effective theory only O_8 contributes.)

5.1.2 Anomalous Dimension matrix and RGE

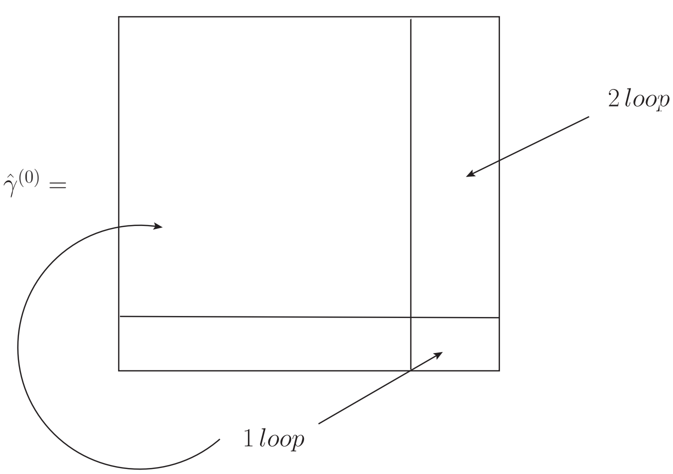
The anomalous dimension matrix $\hat{\gamma}$ can be expanded as

$$\hat{\gamma} = \frac{\alpha_s}{4\pi}\hat{\gamma}^{(0)} + (\frac{\alpha_s}{4\pi})^2\hat{\gamma}^{(1)} + ...$$

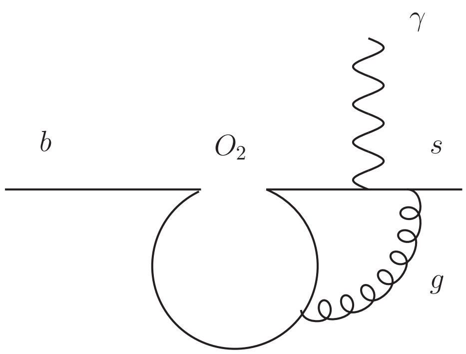
We only need $\hat{\gamma}^0$, reading

$$\hat{\gamma}^{(0)} = \left(\begin{array}{cccccc|cc}
-2 & 6 & 0 & 0 & 0 & 0 & 0 & 3 \\
6 & -2 & -2/9 & 2/3 & -2/9 & 2/3 & 416/81 & 70/27 \\
0 & 0 & -22/9 & 22/3 & -4/9 & 4/3 & -464/81 & 545/27 \\
0 & 0 & 44/9 & 4/3 & -10/9 & 10/3 & 136/81 & 512/27 \\
0 & 0 & 0 & 0 & 2 & -6 & 32/9 & -59/3 \\
0 & 0 & -10/9 & 10/3 & -10/9 & -38/3 & -296/81 & -703/27 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 32/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -32/9 & 28/3
\end{array}\right).$$

$\hat{\gamma}^{(0)} =$



Example for a 2-loop contribution: Contribution to $\gamma^{(0)}_{27}$, i.e.,



$$\boxed{\mu\frac{d}{d\mu}\vec{C}(\mu) = \hat{\gamma}^T\vec{C}(\mu)}$$ RGE for Wilson-coefficients.

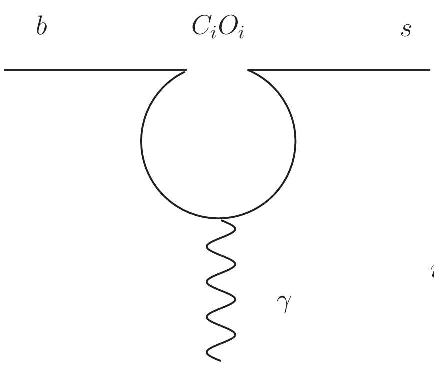
Solve the RGE using $\vec{C}(\mu_W)$ from the pages before as initial condition at $\mu = \mu_W$. Solving the RGE leads to $\vec{C}(\mu_b)$. For explicit expressions see literature, e.g. the review of Buras.

5.1.3 Matrix element $\langle s\gamma|H_{\text{eff}}|b\rangle$

We now have to calculate the matrix elements $\langle s\gamma|C_i(\mu_b)O_i(\mu_b)|b\rangle$ in order α_s^0 precision:

- From the 4-Fermi-operators only O_5 and O_6 contribute.

  a)

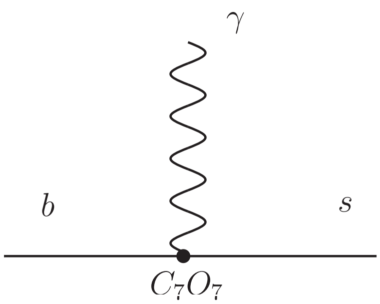


$i = 5, 6$

  Note: $C_5(\mu_b), C_6(\mu_b) \neq 0$ (only $C_5(\mu_W) = C_6(\mu_W) = 0$)

- From O_7:

  b)



It turns out that the results of the loops in a) have the same structure as the tree-level diagram in b).

Can therefore easily summarize both categories a) and b):

$$C_7 \to C_7^{\text{eff}} = C_7 - \frac{1}{3}C_5 - C_6 \,.$$

The amplitude A then reads

$$\boxed{A(b \to s\gamma) = -\frac{4G_F}{\sqrt{2}} V_{tb}V_{ts}^{*} C_7^{\text{eff}}(\mu_b)\langle s\gamma|O_7|b\rangle_{\text{tree}}}$$

Numerically we have for C_7^{eff} at the matching scale $\mu_W = m_W$: $C_7^{\text{eff}}(m_W) = -0.192$. At the low scale μ_b we get (after using the RGE)

$$C_7^{\text{eff}}(10\text{ GeV}) = -0.268\,,$$
$$C_7^{\text{eff}}(5\text{ GeV}) = -0.299\,,$$
$$C_7^{\text{eff}}(2.5\text{ GeV}) = -0.334\,,$$

where μ_b is varied in the typical range $2.5\text{ GeV} \leq \mu_b \leq 10\text{ GeV}$. We see that the dependence on μ_b is large! In LL approximation the μ_b dependence is only in the Wilson-coefficients. This means that the amplitude suffers from a large scale (μ_b) dependence at LL precision!

5.1.4 Result for $BR(b \to s\gamma)$ in various approximations

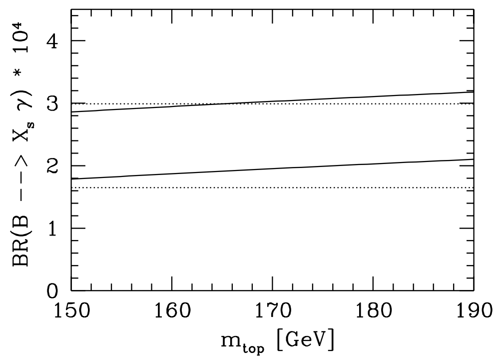


Figure taken from C. G., T. Hurth, D. Wyler, PRD54 (1996) 3350.

The figure shows the branching ratio for $b \to s\gamma$ based on the leading logarithmic (LL) precision. The upper (lower) solid curve is for $\mu_b = m_b/2$ ($\mu_b = 2m_b$). The dotted curves show the CLEO bounds.

Analytically, a term of the form $\sim \alpha_s(m_b) \log \frac{\mu_b}{m_b}$ is responsible for the large scale dependence of the branching ratio $BR(b \to s\gamma)$ at LL precision. This term dies if one systematically calculates the NLL contributions.

NLL-Calculation:

- Matching: $C_i(\mu_W)$ to order α_s^1 precision.
- Anomalous dimensions: $\hat{\gamma} = \frac{\alpha_s}{4\pi}\hat{\gamma}^{(0)} + (\frac{\alpha_s}{4\pi})^2\hat{\gamma}^{(1)}$.

  $\hat{\gamma}^{(1)}$ involves the evaluation of many 3-loop diagrams.
- Matrix elements $\langle s\gamma|O_i(\mu_b)|b\rangle$ are needed to order α_s^1 precision; they are responsible for the cancellation of the term $\sim \alpha_s(m_b) \log \frac{\mu_b}{m_b}$!

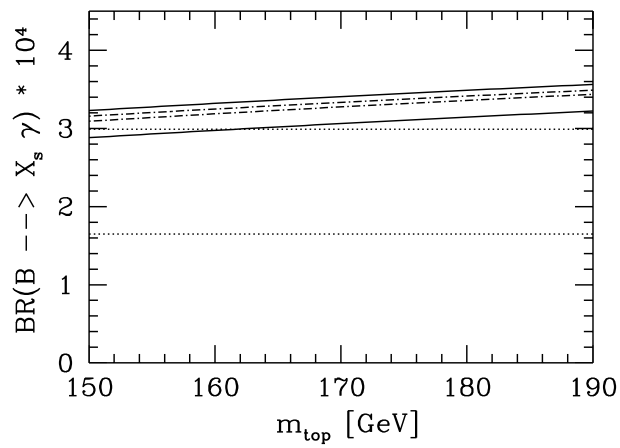
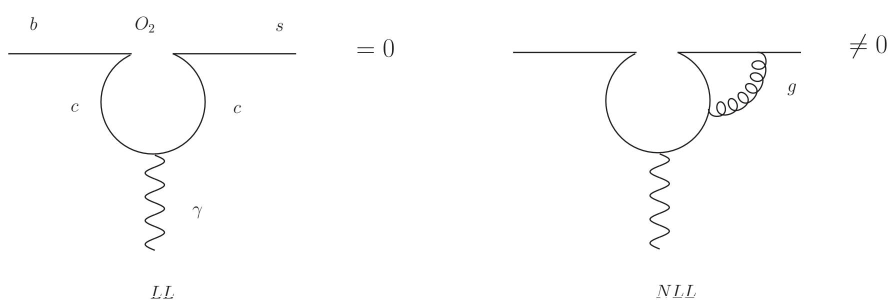


Figure taken from C. G., T. Hurth, D. Wyler, PRD54 (1996) 3350.

The figure shows the branching ratio for $b \to s\gamma$ at next-to-leading logarithmic (NLL) precision. The band defined by the solid curves corresponds to varying μ_b between $m_b/2$ and $2\,m_b$. The dotted curves show the 1σ-CLEO bounds.

In 2001, Gambino and Misiak realized that the NLL result has a large renormalization scheme dependence related to m_c:



m_c enters the first time at the NLL-level, because at LL the diagrams involving m_c are just zero. m_c enters the result at NLL precision. Its definition ($m_c = \bar{m}_c$ or $m_c = m^{\rm pole}$) is unclear.

Numerically however, it makes a difference if one uses $m_c = \bar{m}_c$ or $m_c = m_c^{\rm pole}$. At the level of the BR this leads to an uncertainty of $\sim 11\%$.

<ins>Remedy:</ins> Go to NNLL precision. In this case you have to renormalize m_c in the NLL result; as a consequence you have the renormalization scheme in your hands!

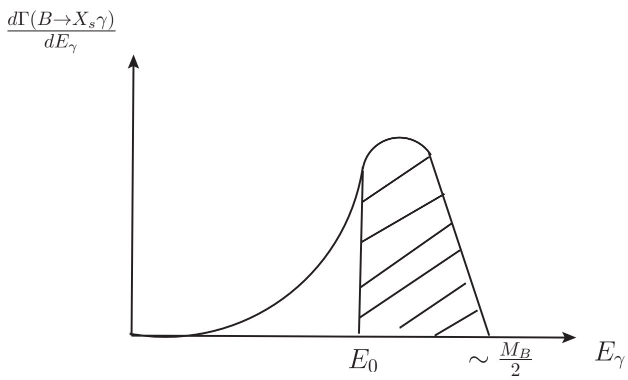
The NNLL calculation was started in 2001; many groups involved; very complicated:

- up to three-loop diagrams for matching
- up to four-loops for certain entries in anomalous dimension matrix
- up to three-loop matrix elements of O_i

NNLL still not completely finished, but the most important contributions are done.

$$\mathrm{BR}(B \to X_s \gamma)^{\mathrm{NNLL}}_{E_\gamma > 1.6\mathrm{GeV}} = (3.36 \pm 0.23) \cdot 10^{-4} \, .$$

PRL-paper, M. Misiak, C. Greub, J. Virto +15 authors, 2015.



E_0 was chosen to be 1.6 GeV. For E below E_0 there is large background.

$BR(B \to X_s \gamma)^{\mathrm{exp}}_{E_\gamma > 1.6\mathrm{GeV}} = (3.43 \pm 0.21 \pm 0.07) \cdot 10^{-4}$ CLEO + BABAR + BELLE averaged.

SM-theory and experiment in good agreement! This is “good” and “bad” at the same time!

good: We have a reliable theory also in the sector of rare B-decays.

bad: Rare decays are potentially very sensitive to extensions of the SM. Instead of top quarks and W-bosons other, non-SM particles could propagate in the loop and modify the branching ratio. $\Rightarrow$ “Only” bounds on new physics can be obtained from this decay.

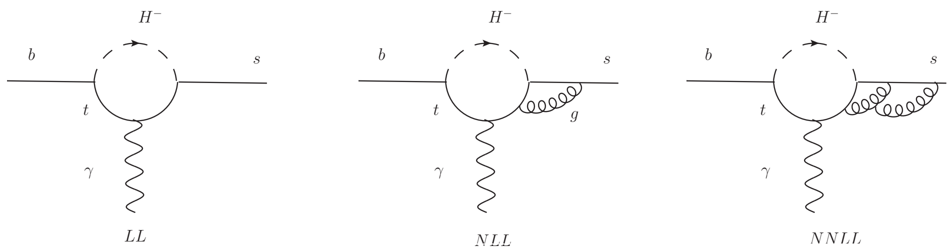
5.1.5 $b \to s\gamma$ in 2HDMs

Not time enough to really do it. Just a few points.

The Higgs sector is enlarged to contain two Higgs-doublets. The physical spectrum contains 5 Higgses: h^0, H^0, A^0, H^+, H^-.

For $b \to s\gamma$ the operator basis is the same as in the SM.

- The only change in the formalism is therefore the matching step:



NLL: Borzumati, C.Greub 1998

NNLL: Misiak et. al 2012

- Anomalous dimension matrix is the same; it only knows about the effective operators, which are the same as in the SM.

- Matrix elements of O_i: same comment.

This process leads to the most stringent constraint on $m_{H^\pm}$ in the 2HDM of type II:

$$m_{H^\pm} \geq 480 \text{ GeV @95\% c.l.}\,.$$

type II: is a particular version of the 2HDM; it is the version which is contained in the MSSM (minimal supersymmetric extension of the SM).

5.2 Process $B \to X_d\gamma$ in SM

$$B \to X_d\gamma \qquad (B \to X_s\gamma)$$

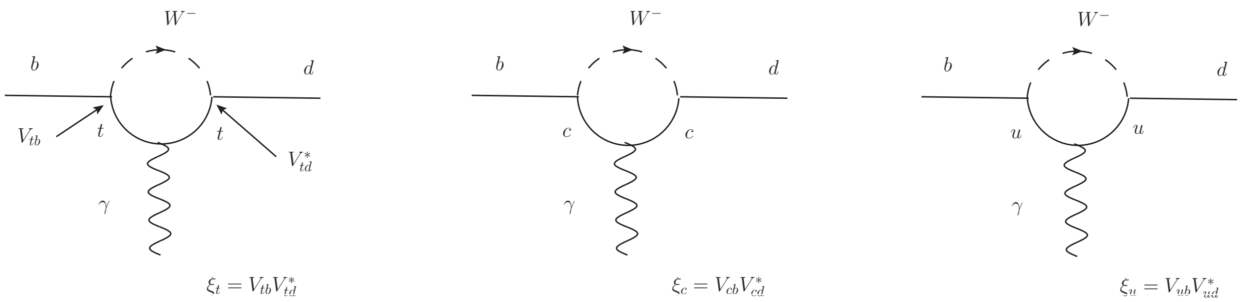
X_d: hadronic matter, no s, no c.

X_s: hadronic matter with strangeness.

Underlying decay at quark-level:

$$b \to d\gamma$$

$b \to d\gamma$ is loop-induced, like $b \to s\gamma$.



Formalism similar as for $b \to s\gamma$. But one important difference:

The u-contribution is not suppressed anymore:

$$|\xi_t| \sim |\xi_c| \sim |\xi_u| \quad (b \to d\gamma) \quad \xi_i = V_{ib}V_{id}^*$$
$$|\lambda_t| \sim |\lambda_c| \gg |\lambda_u| \quad (b \to s\gamma) \quad \lambda_i = V_{ib}V_{is}^*$$

This has an important consequence: $b \to d\gamma$ has quite large $\not{CP}$, while $\not{CP}$ is basically zero in $b \to s\gamma$.

Let us look a bit at this aspect.

$$A(b \to d\gamma) = \xi_t A_t + \xi_c A_c + \xi_u A_u\,; \qquad A_i : \text{loop-functions} \tag{5.2}$$

<u>CP-conjugated process:</u> $\bar{b} \to \bar{d}\gamma$

$$A(\bar{b} \to \bar{d}\gamma) = \xi_t^* A_t + \xi_c^* A_c + \xi_u^* A_u$$

Note that <u>only</u> the CKM factors get complex conjugated.

If the CKM-matrix would be real, then $A(\bar{b} \to \bar{d}\gamma) = A(b \to d\gamma) \Rightarrow$no CP-violation.

In the SM with 2 generations, the CKM-matrix can always be made real by suitable field redefinitions $\to$ no $\not{CP}$

In the SM with 3 generations this is not possible anymore. $\to$ CKM genuinely complex $\to \not{CP}$.

$$a_{CP} \doteq \frac{\Gamma(b \to d\gamma) - \Gamma(\bar{b} \to \bar{d}\gamma)}{\Gamma(b \to d\gamma) + \Gamma(\bar{b} \to \bar{d}\gamma)} \quad \text{CP-rate asymmetry}$$

Suppose that only the first term is present in (5.2).

$$A(b \to d\gamma) = \xi_t A_t; \quad A(\bar{b} \to \bar{d}\gamma) = \xi_t^* A_t$$

Amplitude is CP-violating, but does it lead to $a_{cp} \neq 0$?

$$a_{CP} \sim \frac{|\xi_t A_t|^2 - |\xi_t^* A_t|^2}{|\xi_t A_t|^2 + |\xi_t^* A_t|^2} = 0 \rightarrow \text{ no "visible" } \cancel{CP}.$$

More than one contribution needed!

Two contributions:

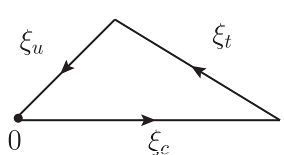
$$a_{CP} \sim \frac{|\xi_t A_t + \xi_c A_c|^2 - |\xi_t^* A_t + \xi_c^* A_c|^2}{|\xi_t A_t + \xi_c A_c|^2 + |\xi_t^* A_t + \xi_c^* A_c|^2}$$
$$a_{CP} \sim \frac{2Re(\xi_t \xi_c^* A_t A_c^*) - 2Re(\xi_t^* \xi_c A_t A_c^*)}{|\xi_t A_t + \xi_c A_c|^2 + |\xi_t^* A_t + \xi_c^* A_c|^2}$$
$$a_{CP} \sim \frac{-4Im(\xi_t \xi_c^*) Im(A_t A_c^*)}{|\xi_t A_t + \xi_c A_c|^2 + |\xi_t^* A_t + \xi_c^* A_c|^2}$$

$a_{CP} \neq 0$ only if:

- A_t, A_c different (strong) phases
- ξ_t, ξ_c different (weak) phases

ξ_t, ξ_c have different phases:

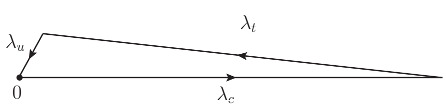
$$\xi_t + \xi_c + \xi_u = 0$$



a_{CP} potentially large

$b \rightarrow s\gamma$: The same formalism, just replace $\xi_i \rightarrow \lambda_i$:

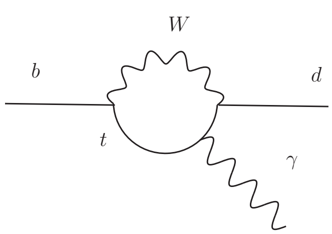
$$\lambda_t + \lambda_c + \lambda_u = 0$$

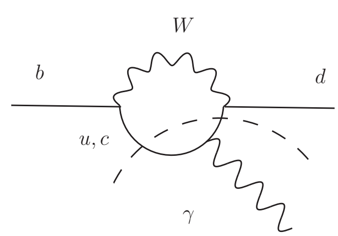


λ_t, λ_c, λ_u are almost relatively real $\rightarrow a_{CP} \approx 0$.

Again $b \rightarrow d\gamma$: To have observable a_{CP}, we also need $Im(A_t A_c^*) \neq 0$; i.e. a (strong) phase difference between A_t and A_c is necessary.

A_t, A_c, A_u denote just the diagrams without the CKM factors.





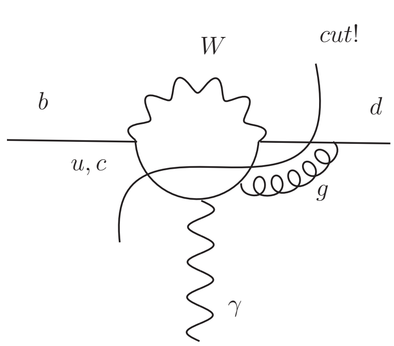
A_t : no physical cut $\rightarrow A_t$ real.

$A_{u,c}$: no cut if γ is on-shell $\rightarrow A_{u,c}$ real.

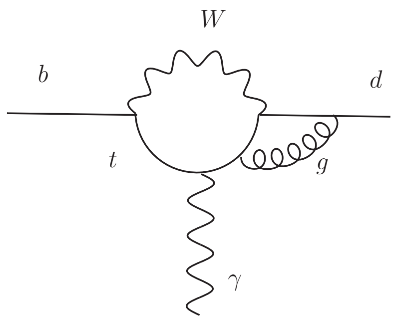
no (strong) phase difference! $\rightarrow a_{CP} = 0$.

Remark: An off-shell photon with $q^2 \geq 4m_c^2$ would provide a phase ($b \rightarrow dl^+l^-$).

Switch on QCD:



Has a cut and hence an imaginary part.



<u>no</u> cut $\rightarrow$ real.

$\Rightarrow A_t$ and A_c have a relative (strong) phase, but only in presence of QCD.

$$a_{CP}(b \to d\gamma) \neq 0$$
$$a_{CP}(b \to d\gamma) = (7\% - 35\%) \quad \text{(should be updated)}$$

The last result is obtained by scanning over the CKM-range as of 1998 (Ali+C.Greub). Today, the allowed CKM-range is much smaller than in 1998, therefore an update should be done.